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Degenerate Diffusions

Initial Value Problems and Local Regularity Theory

European Mathematical Society

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Dedicated to the memory of Björn Dahlberg

Preface

In this monograph we study the initial value problem (Cauchy problem) and the Dirichlet problem for a class of degenerate diffusions modeled on the equation $u_t = \Delta u^m$, $m > 0$, $u \geq 0$. Our approach to these problems is through the use of local regularity estimates and Harnack type inequalities, which yield equicontinuity and hence compactness for families of solutions. The theory is quite complete in the slow diffusion case (porous medium equation) $m > 1$ and in the super-critical fast diffusion case $m_c < m < 1$, where $m_c = (n-2)_+/n$, while problems remain open in the range $m \leq m_c$. In this book we have emphasized the techniques used in the proofs of the results presented, in the hope that they will have a wider scope of applicability beyond the specific problems discussed here. We have also added, at the end of each chapter, a section which discusses further results beyond the main focus of the text and open problems that we find challenging and important.

This book is addressed to both researchers and to graduate students with a good background in analysis and some previous exposure to partial differential equations. Both authors have used with success preliminary versions of the manuscript for second and third year graduate courses in pde.

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Contents

Introduction

We shall study non-negative solutions u of the nonlinear evolution equations

$$\frac{\partial u}{\partial t} = \Delta\varphi(u) \quad x \in \mathbb{R}^n,\ 0 < t < T < +\infty \tag{1}$$

where the nonlinearity φ is assumed to be continuous, increasing, with $\varphi(0) = 0$ and satisfies the growth condition

$$a \leq \frac{u\,\varphi'(u)}{\varphi(u)} \leq a^{-1} \quad \text{for all } u > 0 \tag{2}$$

for some constant $a \in (0, 1)$ and the normalization condition $\varphi(1) = 1$. We shall denote the class of such nonlinearities φ by Γ_a.

The growth condition (2) is a natural generalization of the pure power case $\varphi(u) = u^m$, which is well known in the literature and arises in a number of physical problems. When $m = 1$ it reduces to the linear heat equation. When $m > 1$ equation (1) describes the flow of an isotropic gas through a porous medium (the porous medium equation), cf. [110]. Another application refers to heat radiation in plasmas [136]. When $m < 1$, equation (1) arises in the study of fast diffusions, in particular in models of gas-kinetics [35], [44], in diffusion in plasmas [21], and in thin liquid film dynamics driven by Van der Waals forces [64], [63]. Also it arises in geometry; the case $m = (n-2)/(n+2)$, in dimensions $n > 3$ describes the evolution of a conformal metric by the Yamabe flow [135] and it is related to the Yamabe problem, the case $m = 0$, $n = 2$ describes the Ricci flow on surfaces [77], [53], [133], and the case $m < 0$ in dimension $n = 1$ describes a plane curve shrinking along the normal vector with speed depending on the curvature [72], [68]. For a survey on the porous medium and fast diffusion equations see [111], [127].

The main objective of this book is to present the main results regarding the solvability of the Cauchy problem and the initial Dirichlet problem for equation (1) for a wide class of nonlinearities φ. The local regularity theory for equations of the form (1) will be presented as well. Special emphasis will be given to the various techniques, which although have been developed to study nonlinear equations of the form (1) may be applied to other nonlinear parabolic problems.

In the 1940s D. Widder [132] studied the characterization of the class of all non-negative solutions of the heat equation

$$\frac{\partial u}{\partial t} = \Delta u \quad \text{in } S_T = \mathbb{R}^n \times (0, T]. \tag{3}$$

In this case, the notion of solution is clear: $u \in L^1_{\text{loc}}(S_T)$ and the equation holds in the distribution sense (weak solution). It follows by classical regularity theorems that $u \in C^\infty(S_T)$.

The Widder theory can be expressed as follows. Let u be a non-negative weak solution of the heat equation in the strip S_T. Then:

(A.1) The solution u satisfies the growth condition

$$\sup_{0<t<T/2} \int u(x,t)\, e^{-C\,|x|^2}\, dx \ < \infty \tag{4}$$

where C is an absolute constant.

(A.2) There exists a non-negative Borel measure μ on $\mathbb{R}^n$ such that

$$\lim_{t\downarrow 0} u(\,\cdot, t) = d\mu \quad \text{in } D'(\mathbb{R}^n).$$

We shall call the measure μ the trace of u. Furthermore, the trace μ satisfies the growth condition

$$\int e^{-C\,|x|^2}\, d\mu \ < \infty \tag{5}$$

where C is an absolute constant.

(A.3) The solution u satisfies the pointwise estimate

$$u(x,t) \le C_t(u)\, e^{C\,|x|^2}$$

where C is an absolute constant and $C_t(u)$ depends on u and t.

(A.4) The trace μ determines the solution uniquely; if u, v are two non-negative weak solutions of equation (3) and

$$\lim_{t\downarrow 0} u(\,\cdot, t) = \lim_{t\downarrow 0} v(\,\cdot, t)$$

then $u \equiv v$.

(A.5) For each non-negative Borel measure μ on $\mathbb{R}^n$ satisfying the growth condition (5) there is a non-negative continuous weak solution u of (3) in $S_{T/M}$ with trace μ, and

$$u(x,t) = \frac{C_n}{t^{n/2}} \int e^{-|x-y|^2/4t}\, d\mu(y)$$

for an absolute constant C_n depending only on dimension n.

Let us note that the assumption that the class consists of non-negative solutions is necessary for the Widder theory to hold true, as there are specific examples of oscillating solutions with initial data identically equal to zero which do not satisfy condition (4).

The porous medium equation, i.e. $\varphi(u) = u^m$, $m > 1$, in (1) has been studied extensively and is by now well understood. In fact, by combining the results of Aronson and Caffarelli [8], Bénilan, Crandall, and Pierre [20] and Dahlberg and Kenig [45] one obtains the complete analogue of the Widder theory for this case. Let u be a non-negative continuous distributional solution of the equation

$$\frac{\partial u}{\partial t} = \Delta u^m, \quad m > 1, \ (x, t) \in S_T. \tag{6}$$

Then:

(B.1) The spatial averages of the solution u satisfy the growth condition

$$\sup_{0<t<T} \sup_{R>1} \frac{1}{R^{n+2/(m-1)}} \int_{|x|<R} u(x, t)\, dx \ < \infty.$$

(B.2) The initial trace μ exists; for any continuous distributional solution u of (1) there exists a Borel measure μ such that

$$\lim_{t \downarrow 0} u(\,\cdot, t) = d\mu \quad \text{in } D'(\mathbb{R}^n)$$

and satisfies the growth estimate

$$\sup_{R>1} \frac{1}{R^{n+2/(m-1)}} \int_{|x|<R} d\mu \ < \infty. \tag{7}$$

(B.3) The solution u satisfies the pointwise growth estimate

$$u^{m-1}(x, t) \le C_t(u)\,(1 + |x|^2) \quad \text{for } t \in (0, T/2)$$

where $C_t(u) = C_T\,(u(0, T))\, t^{-\lambda}$, $\lambda = n/(2 + n(m-1))$ as $t \downarrow 0$.

(B.4) The trace μ determines the solution uniquely; if u, v are two non-negative continuous distributional solutions and

$$\lim_{t \downarrow 0} u(\,\cdot, t) = \lim_{t \downarrow 0} v(\,\cdot, t)$$

then $u \equiv v$.

(B.5) For every measure μ on $\mathbb{R}^n$ satisfying (7) there exists a non-negative continuous distributional solution of (6) with trace μ satisfying (7).

Let us note that the assumption on the continuity of u is not essential, due to the result of Dahlberg and Kenig in [49] where it is shown that if $u \in L^m_{\text{loc}}(\Omega)$, $u \ge 0$ and $\partial u/\partial t = \Delta u^m$ in $D'(\Omega)$, then u is continuous.

In order to extend the above results from the porous medium equation $\varphi(u) = u^m$, $m > 1$ to the case of equation (1) we consider the class $\mathcal{S}_a$ of nonlinearities φ, corresponding to *slow diffusion*, which is defined by the following conditions:

(i) $\varphi\colon [0, \infty) \to \mathbb{R}$ is continuous, strictly increasing with $\varphi(0) = 0$;

(ii) there exist $a \in (0,1)$ such that for any $u > 0$

$$a \le \frac{u\,\varphi'(u)}{\varphi(u)} \le \frac{1}{a} \quad \text{(polynomial growth);}$$

(iii) there exists $u_0 > 0$ such that and for any $u \ge u_0$

$$1 + a \le \frac{u\,\varphi'(u)}{\varphi(u)} \quad \text{(super-linearity);}$$

(iv) $u_0 = 1$ and $\varphi(1) = 1$ (normalization).

The condition (iv) is only technical and imposed to normalize the class $\mathcal{S}_a$. Conditions (ii) and (iv) imply the pointwise polynomial growth condition

$$u^{1/a} \le \varphi(u) \le u^a \quad \text{for } 0 \le u \le 1$$

while conditions (iii) and (iv) imply the pointwise super-linear polynomial growth condition

$$u^{1+a} \le \varphi(u) \le u^{1/a} \quad \text{for } u \ge 1.$$

It is clear that the porous medium equation $\varphi(u) = u^m$, $m > 1$, belongs to $\mathcal{S}_a$. However, the super-linear growth on $\varphi(u) \in \mathcal{S}_a$ is only assumed for large values of u.

The Widder theory in this case can be described as follows: by hypothesis $\psi(u) = \varphi(u)/u$, for $u \ge 1$ is an increasing function, so that we may define $\Lambda(u) \equiv \psi^{-1}(u)$. Thus if u is a non-negative continuous weak solution of (1) then the following holds:

(C.1) Growth condition (B.1) holds with $\Lambda(R^2)$ instead of $R^{2/(m-1)}$.

(C.2) Condition (B.2) holds with $\Lambda(R^2)$ instead of $R^{2/(m-1)}$ in the growth estimate (7) on the initial trace.

(C.3) The pointwise estimate (B.3) holds with $\psi(u)$ instead of u^{m-1}.

(C.4) and (C.5) are similar to (B.4) and (B.5).

Following the ideas in [49] one can remove the assumption on the continuity of u in the case where φ is convex. For general φ in the class $\mathcal{S}_a$ an analogous result remains an open problem.

Consider next equation (1) with nonlinearity φ which is either a power $\varphi(u) = u^m$ with $(n-2)/n < m < 1$ or, more generally, it belongs to the class $\mathcal{F}_a$, corresponding to super-critical fast diffusion, which is defined by the following conditions:

(i) $\varphi\colon [0,\infty) \to \mathbb{R}$ is continuous, strictly increasing with $\varphi(0) = 0$;

(ii) there exists $a \in (0,1)$ such that for any $u > 0$

$$a \le \frac{u\,\varphi'(u)}{\varphi(u)} \le \frac{1}{a} \quad \text{(polynomial growth);}$$

(iii) there exists $u_0 > 0$ such that for $u \geq u_0$

$$\frac{n-2}{n} + a \leq \frac{u\,\varphi'(u)}{\varphi(u)} \leq 1 - a \quad \text{(sublinearity);}$$

(iv) $u_0 = 1$ and $\varphi(1) = 1$ (normalization).

Conditions (ii) and (iv) imply the pointwise polynomial growth condition

$$u^{1/a} \leq \varphi(u) \leq u^a \quad \text{for } 0 \leq u \leq 1$$

while conditions (iii) and (iv) imply the pointwise sublinear (since $1 - a < 1$) and super-critical (since $(n-2)/n + a > (n-2)/n$) polynomial growth condition

$$u^{\frac{n-2}{n}+a} \leq \varphi(u) \leq u^{1-a} \quad \text{for } u \geq 1.$$

The above growth conditions generalize the fast diffusion equation $\varphi(u) = u^m$, in the super-critical range of exponents $(n-2)/n < m < 1$, which in particular belongs to the class $\mathcal{F}_a$.

It follows by the results of Herrero and Pierre [81], and Dahlberg and Kenig [47] that in the fast diffusion case no growth conditions need to be imposed on the initial trace for existence, as described next:

(D.1) For any non-negative continuous distributional solution u of (1), there exists a unique locally finite Borel measure μ on $\mathbb{R}^n$ such that

$$\lim_{t \downarrow 0} \int_{\mathbb{R}^n} u(x,t)\,\psi(x)\,dx = \int_{\mathbb{R}^n} \psi(x)\,d\mu(x)$$

for all $\psi \in C_0^\infty(\mathbb{R}^n)$.

(D.2) The trace μ determines the solution uniquely: if u, v are two non-negative continuous distributional solutions and

$$\lim_{t \downarrow 0} u(\,\cdot\,,t) = \lim_{t \downarrow 0} v(\,\cdot\,,t)$$

then $u \equiv v$.

(D.3) For any locally finite Borel measure μ on $\mathbb{R}^n$ there exists a continuous distributional solution u of (1) in $S_\infty = \mathbb{R}^n \times (0,\infty)$ with trace μ.

(D.4) For any non-negative continuous distributional solution u of (1) in S_T, there exists a non-negative continuous distributional solution $\hat{u}$ of (1) in S_∞ with $u = \hat{u}$ in S_T.

The super-critical assumption (iii) is essential for the theory described above. Indeed, in the sub-critical case $m \leq (n-2)/n$ the analogues of the above results do not hold true. In particular, there is no continuous distributional solution of equation

$u_t = \Delta u^m$, $m \leq (n-2)/n$ with initial data the Dirac mass. We refer the reader to Section 3.3 for details.

One particularly interesting case of fast diffusion is the case where $\varphi(u) = \log u$, corresponding to the limiting case of $\varphi(u) = u^m$, when $m \to 0$. It has been shown by Esteban, Rodríguez and Vazquez in [69], and Daskalopoulos and del Pino in [52] that a strong non-uniqueness phenomenon takes place in this case. In the critical dimension $n = 2$ this phenomenon is related to the topological properties of solutions to the Ricci flow, corresponding to evolving metrics on compact surfaces, non-compact surfaces and orbifolds. We refer the reader to Section 3.2 of Chapter 3 for the details on the solvability and well-posedness of the Cauchy problem for the logarithmic fast diffusion equation.

A brief outline of the contents of the book is as follows.

In Chapter 1 we shall collect a series of preliminary, yet very important results, concerning continuous distributional solutions of equation (1). These results will be used throughout the book. We emphasize the a priori L^∞ bounds, the Harnack inequality for solutions of slow diffusion, and the equicontinuity of solutions.

Chapter 2 deals with the solvability of the Cauchy problem for equation (1) in the slow diffusion case $\varphi \in \mathcal{S}_a$. We shall provide a complete characterization of non-negative weak solutions of (2.0.1) in terms of their initial condition, showing in particular the results (B.1)–(B.5) and their extensions (C.1)–(C.5).

The first part of Chapter 3 is devoted to the solvability of the Cauchy problem for equation (1) in the super-critical fast diffusion case $\varphi \in \mathcal{F}_a$. We shall present a theory which completely classifies the class of continuous weak solutions of (3.1.1) in terms of their initial condition, showing in particular (D.1)–(D.4). The second part of this chapter is devoted to the Cauchy problem for the logarithmic fast diffusion equation $u_t = \Delta \log u$, in dimensions $n \geq 2$. We give special emphasis to the critical case $n = 2$, where many interesting phenomena can be observed with important geometric applications. We show that in this case a strong non-uniqueness phenomenon takes place and establish the existence a continuum of solutions with a given initial data. In the last section of this chapter we comment on the solvability and well-posedness of equation $u_t = \Delta u^m$ in the sub-critical case $0 < m \leq (n-2)/n$ as well as in the super-fast diffusion case $m < 0$.

In Chapter 4 we study the class of non-negative strong solutions of the initial Dirichlet problem for equation (1) on $D \times (0, \infty)$, $D \subset \mathbb{R}^n$ open bounded, in the slow diffusion case $\varphi \in \mathcal{S}_a$. We establish the existence of an exceptional solution α with infinite initial data. We then show that any other strong solution of the initial Dirichlet problem is uniquely characterized by its initial traces μ and λ, namely non-negative Borel measures that are supported on D and ∂D respectively. We also study in this chapter the initial Dirichlet problem for equation (1) in the pure power fast diffusion case, $\varphi(u) = u^m$, in the range $(n-2)_+/n < m < 1$.

Our last Chapter 5 is devoted to the study of the regularity properties of weak solutions to the porous medium equation $u_t = \Delta u^m$, $m > 1$. We establish that weak solutions are continuous.

The last section in each chapter is devoted to a brief summary of further known results as well as several open problems related to the theory presented.

Denote by Γ_a the class of nonlinearities φ which satisfy the following conditions:

(i) $\varphi\colon [0,\infty) \to \mathbb{R}$ is a continuous non-negative function with $\varphi(0) = 0$;

(ii) there exist a constant $a \in (0,1)$ such that

$$a \leq \frac{u\,\varphi'(u)}{\varphi(u)} \leq a^{-1} \quad \text{for all } u > 0;$$

(iii) $\varphi(1) = 1$ (normalization).

For future reference, we close the Introduction with the statement of three results, proved in following chapters, which will play a fundamental role throughout the book.

The following compactness result due to P. Sacks [118] will be proved in Section 1.5 of Chapter 1, and will be used extensively throughout the book.

Theorem H.1. *Let $\{u_k\}$ be a sequence of continuous non-negative distributional solutions of the equation $\partial u/\partial t = \Delta\varphi(u)$ in R, with $\varphi \in \Gamma_a$. If $\{u_k\}$ is uniformly bounded in R, then the u_k's are equicontinuous in S.*

In the case that $\varphi \in \mathcal{S}_a$, the following stronger result due to DiBenedetto and Friedman [67] holds: *Let u be a non-negative continuous distributional solution of the equation $u_t = \Delta\varphi(u)$ in a compact domain $R \subset \mathbb{R}^n \times (0,\infty)$, with $\varphi \in \mathcal{S}_a$. Then*

$$\left|u(x,t) - u(x',t')\right| \leq C(M)\ \left\{|x - x'| + |t - t'|\right\}^{\alpha}$$

for any $(x,t), (x',t') \in S \subset\subset R$, where $\alpha = \alpha(a,n)$ and the constant M is given by $M = \sup_t \int_R u(x,t)\,dx$.

In other words, continuous solutions in R are Hölder continuous in S. This result is sharp, i.e. examples show that u need not be more regular than Hölder continuous. To see this consider the Barenblatt self-similar solution of the Cauchy problem

$$\begin{aligned} u_t &= \Delta u^m \quad \text{in} \mathbb{R}^n \times \mathbb{R}^+, \\ u(x,0) &= \delta(0) \quad\ x \in \mathbb{R}^n, \end{aligned}$$

which has the explicit form

$$B(x,t) = t^{-\alpha} \left[\left(M - k\,\frac{|x|^2}{t^{\beta}}\right)_+\right]^{\frac{1}{m-1}}$$

with

$$\alpha = \frac{n}{n\,(m-1)+2}, \qquad \beta = \frac{1}{n(m-1)+2}$$

and M, k specific constants which depend only on m, n. It is clear that for $m > 2$, the solution $B(x,t)$ is Hölder continuous with exponent $\alpha \leq \alpha_0(m,n)$ but not Lipschitz.

We shall use the following weaker version of (C.5) which will be proved in Section 1.6 of Chapter 1:

Theorem H.2. *Let $f \in L^1(\mathbb{R}^n)$ be a non-negative function. Then there exists a unique non-negative continuous distributional solution of the equation* (1) *in $\mathbb{R}_+^{n+1}$ such that*

$$\sup_{t>0} \int u(x,t)\,dx \leq \int f(x)\,dx$$

and

$$\lim_{t \downarrow 0} \|u(\cdot,t) - f(x)\|_{L^1(\mathbb{R}^n)} = 0.$$

Furthermore if f is radially decreasing so is $u(\cdot,t)$, for each $t > 0$.

The following uniqueness result due to M. Pierre [113] will be proved in Section 2.4 of Chapter 2:

Theorem H.3. *If u_1 and u_2 are continuous non-negative distributional solutions of the equation* (1) *with $\varphi \in \Gamma_a$ such that*

$$\sup_{t>0} \int (u_1(x,t) + u_2(x,t))\,dx < \infty$$

with

$$u_1, u_2 \in L^\infty(\mathbb{R}^n \times [\tau, \infty)) \quad \textit{for each } \tau > 0$$

and

$$\lim_{t \downarrow 0} u_1(\cdot,t) = \lim_{t \downarrow 0} u_2(\cdot,t) \quad \textit{in } D'(\mathbb{R}^n)$$

then $u_1 = u_2$.

Chapter 1
Local regularity and approximation theory

We shall collect and prove in this chapter various preliminary results which will be used throughout the coming chapters. All the results in this chapter will be concerned with non-negative solutions of equation

$$\frac{\partial u}{\partial t} = \Delta\varphi(u) \tag{1.0.1}$$

where the nonlinearity φ is assumed to belong to the class Γ_a defined by the following conditions:

(i) $\varphi\colon [0, \infty) \to \mathbb{R}$ is a continuous non-negative function with $\varphi(0) = 0$;

(ii) there exists a constant $a \in (0, 1)$ such that

$$a \leq \frac{u\,\varphi'(u)}{\varphi(u)} \leq a^{-1} \quad \text{for all } u > 0; \tag{1.0.2}$$

(iii) $\varphi(1) = 1$ (normalization).

For some of the results φ will be assumed to belong to the subclass $\mathcal{S}_a$ of Γ_a, corresponding to slow diffusion, which is defined by the additional condition:

(iv) for the same constant $a \in (0, 1)$, φ satisfies the additional growth condition (super-linearity)

$$1 + a \leq \frac{u\,\varphi'(u)}{\varphi(u)} \quad \text{for all } u \geq 1. \tag{1.0.3}$$

1.1 Maximum principle and approximation

In this section we shall establish a preliminary version of the maximum principle. This will be used to prove an approximation procedure which justifies the a priori estimates in the coming sections.

For $x_0 \in \mathbb{R}^n$ and $r > 0$, set $B = B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$, and for $\tau_1 < \tau_2$, let $Q = B \times (\tau_1, \tau_2)$. Denote by $\partial_p Q$ the parabolic boundary of Q, i.e. $\partial_p Q = \partial Q \setminus (B \times \{\tau_2\})$.

We shall consider the following boundary value problem (BVP):

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \Delta\varphi(u) && \text{in } Q, \\ u &= g && \text{in } \partial_p Q, \end{aligned}\right\} \tag{1.1.1}$$

where $g \in C(\partial_p Q)$ with $g \geq 0$ is given.

Definition. A function $u(x,t)$ is said to be a weak solution of (1.1.1) if $u \geq 0$, $u \in C\big([\tau_1, \tau_2] : L^1(B)\big) \cap L^\infty(Q)$ and u satisfies the integral identity

$$\iint_Q \left[\varphi(u)\, \Delta\eta + u\, \frac{\partial \eta}{\partial t} \right] dxdt$$
$$= \int_{\tau_1}^{\tau_2} \int_{\partial B} \varphi(g) \frac{\partial \eta}{\partial n}\, d\sigma dt + \int_B u(x,\tau_2)\, \eta(x,\tau_2)\, dx - \int_B g(x,\tau_1)\, \eta(x,\tau_1)\, dx \tag{1.1.2}$$

for any $\eta \in C^\infty(\bar{Q})$ vanishing on $\partial B \times [\tau_1, \tau_2]$.

Here $\partial/\partial n$ denotes the exterior normal derivative on ∂B and σ denotes the surface measure on ∂B.

Remarks. 1. The condition $u \in C\big([\tau_1, \tau_2] : L^1(B)\big) \cap L^\infty(Q)$ guarantees that (1.1.2) makes sense.

2. It will be proved later (Proposition 1.1.3) that if u is a weak solution on the cylinder Q then it is also a weak solution on any cylinder $Q' \subset Q$ with the appropriate boundary condition.

We have the following comparison principle for weak solutions of (1.1.1).

Theorem 1.1.1 ([12] [45]). *Let $g_1, g_2 \in C(\partial_p Q)$ and let u_1, u_2 be weak solutions of the BVP (1.1.1) with boundary values g_1 and g_2 respectively. If $0 \leq g_1 \leq g_2$ on $\partial_p Q$, then $u_1 \leq u_2$ in Q.*

Proof. Fix $s \in (\tau_1, \tau_2]$ and consider the cylinder $Q(s) = B \times [\tau_1, s]$. Set $b = u_1 - u_2$ and consider η a non-negative test function, with $\eta \equiv 0$ and $\partial\eta/\partial n \leq 0$ on $\partial_l Q(s)$, where $\partial_l Q(s) = \partial B \times [\tau_1, s]$ denotes the lateral boundary of $Q(s)$. Using the integral identity (1.1.2) it follows that

$$\int_B b(x,s)\, \eta(x,s)\, dx \leq \iint_{Q(s)} b \left[\frac{\partial \eta}{\partial t} + A\, \Delta\, \eta \right] dxdt \tag{1.1.3}$$

with $A \geq 0$ given by

$$A = \begin{cases} \frac{\varphi(u_1) - \varphi(u_2)}{u_1 - u_2} & u_1 \neq u_2, \\ \varphi'(u_1) & u_1 = u_2. \end{cases}$$

Fix now a function $h \in C_0^\infty(B)$, $h \geq 0$. For $E \in C^\infty(\bar{Q})$ with $E > 0$ on $\bar{Q}$, let $\eta \equiv S(E)$ be the solution of the following (backward) BVP on $Q(s)$:

$$\left. \begin{aligned} \partial\eta/\partial t + E\, \Delta\eta = 0 \quad & \text{in } Q(s), \\ \eta = 0 \quad & \text{on } \partial_l Q(s), \\ \eta(x,s) = h(x) \quad & x \in B. \end{aligned} \right\} \tag{1.1.4}$$

By the classical linear theory we have that $\eta \in C^\infty(\bar{Q})$, $\eta \geq 0$ in $Q(s)$ and $\partial\eta/\partial n \leq 0$ on $\partial_l Q(s)$. Moreover

$$\iint_{Q(s)} E\,(\Delta\eta)^2\,dxdt \leq \frac{1}{2}\int_B |\nabla h|^2\,dx. \tag{1.1.5}$$

To prove (1.1.5) we multiply the equation in (1.1.4) by $\Delta\eta$ and integrate in $Q(s)$. Thus

$$\begin{aligned}
\iint_{Q(s)} E\,(\Delta\eta)^2 dxdt &= -\iint_{Q(s)} \Delta\eta\,\frac{\partial\eta}{\partial t}\,dxdt \\
&= -\int_B h\,\Delta h\,dx + \int_B \eta(x,\tau_1)\Delta\eta(x,\tau_1)\,dx + \iint_{Q(s)} \eta\,\frac{\partial(\Delta\eta)}{\partial t}\,dxdt \\
&= \int_B |\nabla h|^2\,dx - \int_B |\nabla\eta|^2(x,\tau_1)\,dx + \iint_{Q(s)} \eta\,\frac{\partial(\Delta\eta)}{\partial t}\,dxdt \\
&\leq \int_B |\nabla h|^2 dx - \iint_{Q(s)} E\,(\Delta\eta)^2 dxdt
\end{aligned}$$

because

$$\iint \eta\,\frac{\partial(\Delta\eta)}{\partial t}\,dxdt = -\iint \eta\,\Delta(E\,\Delta\eta)\,dxdt = -\iint E\,(\Delta\eta)^2\,dxdt$$

since $\Delta\eta \equiv 0$ on the lateral sides ($E\,\Delta\eta = -\partial\eta/\partial t$ and $E > 0$).

An approximation argument shows that if we only assume that E satisfies $0 < c < E < C$ in $Q(s)$ then there exists $\eta \equiv S(E)$ with $\partial\eta/\partial t + E\,\Delta\eta = 0$ in $Q(s)$, and

$$\int_B b(x,s)h(x)dx \leq \iint_{Q(s)} b\left[\frac{\partial\eta}{\partial t} + A\,\Delta\eta\right]dxdt.$$

Choose now a sequence $\varepsilon_k \downarrow 0$ and define

$$\alpha_k = \frac{|\varphi(u_1) - \varphi(u_2)|}{\varepsilon_k + |u_1 - u_2|}, \qquad A_k = \alpha_k + \varepsilon_k,$$

and

$$\eta_k = S(A_k).$$

Thus we obtain

$$\begin{aligned}
I = \int_B b(x,s)h(x)dx &\leq \iint_{Q(s)} b\left[\frac{\partial\eta_k}{\partial t} + A\,\Delta\eta_k\right]dxdt \\
&= \iint_{Q(s)} b(A - A_k)\Delta\eta_k\,dxdt
\end{aligned}$$

and from (1.1.5) it follows that

$$\begin{aligned}
(\max(I,0))^2 &\leq \left(\iint_{Q(s)} A_k\,(\Delta\eta_k)^2\right)\left(\iint_{Q(s)} \frac{(A - A_k)^2}{A_k}\,b^2\,dxdt\right) \\
&\leq \left(\frac{1}{2}\int_B |\nabla h|^2\,dx\right)\left(\iint_{Q(s)} \frac{(A - A_k)^2}{A_k}\,b^2\,dxdt\right).
\end{aligned} \tag{1.1.6}$$

Next, observe that

$$(A - A_k)^2 = (A - \alpha_k - \varepsilon_k)^2 \le (A - \alpha_k)^2 + \varepsilon_k^2$$

and

$$A - \alpha_k = \frac{\varepsilon_k A}{\varepsilon_k + |u_1 - u_2|}.$$

Hence

$$\begin{aligned} \frac{(A - \alpha_k)^2 b^2}{A_k} &= \frac{\varepsilon_k^2 A^2 b^2}{A_k(\varepsilon_k + |u_1 - u_2|)^2} \\ &\le \frac{\varepsilon_k^2 A\,|b|}{\varepsilon_k + |u_1 - u_2|} \\ &\le \varepsilon_k\,|\varphi(u_1) - \varphi(u_2)| \le C\,\varepsilon_k \end{aligned}$$

since $A_k = \alpha_k + \varepsilon_k \ge \alpha_k$ and $u_1, u_2 \in L^\infty(Q)$. Also

$$\frac{\varepsilon_k^2 b^2}{A_k} \le \varepsilon_k b^2 \le C\,\varepsilon_k.$$

Therefore, letting $k \to \infty$ in (1.1.6) it follows that

$$\int_B h(x) b(x,s)\,dx \le 0$$

for any $h \in C_0^\infty(B)$, $h \ge 0$. Thus $b(x,s) = u_1(x,s) - u_2(x,s) \le 0$, finishing the proof. □

The following approximation result will be used extensively in the sequel.

Corollary 1.1.2. *Let $g \in C(\partial_p Q)$ be non-negative, and suppose u is a weak solution of the BVP (1.1.1). Assume also that u is continuous in $\overline{Q}$ and that $G_k \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ have been chosen so that $g_k = G_k|_{\partial_p Q}$ are strictly positive, $g \le g_{k+1} \le g_k$, and that g_k converges to g uniformly. Let $\varphi \in C^\infty([0,\infty))$, φ_k belonging to the class Γ_a defined above such that $\varphi_k \to \varphi$ uniformly on compact subsets of $[0,\infty)$. Let u_k solve $\partial u_k/\partial t = \Delta\varphi_k(u_k)$ in Q with $u_k = g_k$ on $\partial_p Q$. Then each $u_k \in C^\infty(\overline{Q})$ and u_k converges to u uniformly on compact subsets of Q.*

Proof. Since $0 < \inf_{\partial_p Q} g_k = m_k \le u_k \le M_1 = \max_{\partial_p Q} g$, by the maximum principle, the existence and regularity of u_k follows by combining the classical linear theory for parabolic equation with the Picard method (see [103]). From Theorem H.1 stated in the introduction (i.e. the equicontinuity result due to P. Sacks [118]) it follows that there exists w uniform limit of a subsequence of $\{u_k\}$. Then w is a weak solution of (1.1.1). Hence, by Theorem 1.1.1, $w = u$ which yields the corollary. □

Proposition 1.1.3 ([8]). *Let u be a non-negative continuous solution of $\partial u/\partial t = \Delta\varphi(u)$ in $D'(\Omega)$, $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}$. Then for any $Q \subset\subset \Omega$, u is a weak solution of the BVP (1.1.1) with boundary value $g = u|_{\partial_p Q}$.*

Proof. It suffices to show that u satisfies the integral identity (1.1.2) in Q. Let $Q = B_r \times (\tau_1, \tau_2)$ and let η be a $C^\infty(\bar{Q})$ function with $\eta \equiv 0$ on $\partial_l Q = \partial B_r \times (\tau_1, \tau_2)$. For each $\varepsilon \in (0, r)$ and $\theta \in [0, \varepsilon)$ let $\Psi_{\varepsilon\theta}$ denote the continuous function such that $\Psi_{\varepsilon\theta}(x) = \Psi_{\varepsilon\theta}(|x|)$ with

$$\Psi_{\varepsilon\theta}(\rho) = \begin{cases} 1 & \text{for } 0 \le \rho \le r - \varepsilon, \\ 0 & \text{for } \rho \ge r - \theta, \end{cases}$$

and

$$\Delta\Psi_{\varepsilon\theta} = |x|^{-n+1} \left(|x|^{n-1} \ \Psi_{\varepsilon\theta} \right)' = 0 \quad \text{in } B_{r-\theta} \setminus \bar{B}_{r-\varepsilon}.$$

One can compute $\nabla\Psi_{\varepsilon\theta}$ explicitly as

$$\nabla\Psi_{\varepsilon\theta} = \begin{cases} -W_{\varepsilon\theta} \cdot \frac{x}{|x|^n} & \text{for } x \in B_{r-\theta} \setminus \bar{B}_{r-\varepsilon}, \\ 0 & \text{otherwise,} \end{cases}$$

where for $n > 2$

$$W_{\varepsilon\theta} = \frac{(n-2)\,(r-\theta)\,(r-\varepsilon)^{n-2}}{(r-\theta)^{n-2} - (r-\varepsilon)^{n-2}}.$$

As a distribution in $\mathbb{R}^n$, $\Delta\Psi_{\varepsilon\theta}$ is the signed measure

$$(\Delta\Psi_{\varepsilon\theta})(dx) = W_{\varepsilon\theta} \left\{ \delta_{r-\theta}(|x|) - \delta_{r-\varepsilon}(|x|) \right\} dw$$

where dw denotes the surface element on the unit ball in $\mathbb{R}^n$, and $\delta_\lambda(|x|)$ denotes the Dirac measure concentrated at $|x| = \lambda$. Similar expressions can be derived for $n = 1, 2$.

Let K_ν be a C^∞ radially symmetric function with support in $B_\nu(0)$, $K_\nu \ge 0$ and $\int K_\nu dx = 1$. Define

$$\Psi^\nu_{\varepsilon\theta}(x) = \int K_\nu(x - \xi)\, \Psi_{\varepsilon\theta}(\xi) d\xi.$$

In particular, if ν is sufficiently small $\Delta\Psi^\nu_{\varepsilon\theta}$ is supported in two disjoint annular regions: one containing $\partial B_{r-\varepsilon}$ and the other containing $\partial B_{r-\theta}$. A simple approximation argument (in the time variable) shows that

$$\iint_\Omega \left[\varphi(u)\, \Delta(\eta\, \Psi^\nu_{\varepsilon\theta}) + u\, \Psi^\nu_{\varepsilon\theta} \frac{\partial\eta}{\partial t} \right] dxdt$$
$$= \int_B (u\eta\Psi^\nu_{\varepsilon\theta})(x, \tau_2)\, dx - \int_B (u\eta\Psi^\nu_{\varepsilon\theta})(x, \tau_1)\, dx.$$

When ν tends to zero in the above identity we obtain

$$\begin{aligned}
\iint_Q \left[\varphi(u)\Psi_{\varepsilon\theta}\Delta\eta + u\,\Psi_{\varepsilon\theta}\frac{\partial\eta}{\partial t}\right] dxdt \\
+ 2\int_{\tau_1}^{\tau_2}\int_{B_{r-\theta}\setminus B_{r-\varepsilon}} \varphi(u)\,\nabla\Psi_{\varepsilon\theta}\cdot\nabla\eta\,dxdt \\
+ W_{\varepsilon\theta}\int_{\tau_1}^{\tau_2}\left\{\int_{\partial B_{r-\theta}}\varphi(u)\,\eta\,d\sigma - \int_{\partial B_{r-\varepsilon}}\varphi(u)\,\eta\,d\sigma\right\} dt \\
= \int_B (u\eta\Psi_{\varepsilon\theta})(x,\tau_2)\,dx - \int_B (u\eta\Psi_{\varepsilon\theta})(x,\tau_1)\,dx.
\end{aligned}$$

Now we let $\theta \to 0$. Since $\eta \equiv 0$ on $\partial B_r \times [\tau_1, \tau_2]$ it follows that

$$\begin{aligned}
\iint_Q \left[\varphi(u)\Psi_{\varepsilon 0}\Delta\eta + u\Psi_{\varepsilon 0}\frac{\partial\eta}{\partial t}\right] dxdt + I_\varepsilon + J_\varepsilon \\
\equiv \int_B [\,(u\eta\Psi_{\varepsilon 0})(x,\tau_2) - (u\eta\Psi_{\varepsilon 0})(x,\tau_1)\,]dx
\end{aligned} \tag{1.1.7}$$

where

$$I_\varepsilon = 2\int_{\tau_1}^{\tau_2}\int_{B_r - B_{r-\varepsilon}} \varphi(u)\nabla\Psi_{\varepsilon 0}\nabla\eta\,dxdt$$

and

$$J_\varepsilon = -W_{\varepsilon 0}\int_{\tau_1}^{\tau_2}\int_{\partial B_{r-\varepsilon}} \varphi(u)\eta\,d\sigma.$$

Using polar coordinates we find

$$I_\varepsilon = -2W_{\varepsilon 0}\int_{\tau_1}^{\tau_2}\int_{r-\varepsilon}^{r}\left(\int_{S^{n-1}}\varphi(u)\frac{\partial\eta}{\partial n}|_{|x|=\rho}d\sigma\right)d\rho dt.$$

Observe that $\lim_{\varepsilon\downarrow 0} W_{\varepsilon 0} = r^{n-1}$. Therefore

$$\lim_{\varepsilon\downarrow 0} I_\varepsilon = -2r^{n-1}\int_{\tau_1}^{\tau_2}\int_{S^{n-1}}\varphi(u)\frac{\partial\eta}{\partial n}|_{|x|=r}\,d\sigma dt.$$

On the other hand, since $\eta \equiv 0$ for $|x| = r$, we have

$$J_\varepsilon = W_{\varepsilon 0}\int_{\tau_1}^{\tau_2}\int_{S^{n-1}}\varphi(u)\left\{\eta|_{|x|=r} - \eta|_{|x|=r-\varepsilon}\right\} d\sigma dt$$

implying that

$$\lim_{\varepsilon\downarrow 0} J_\varepsilon = r^{n-1}\int_{\tau_1}^{\tau_2}\int_{S^{n-1}}\varphi(u)\,\frac{\partial\eta}{\partial n}|_{|x|=r}d\sigma dt.$$

Letting $\varepsilon \downarrow 0$ in (1.1.7) we obtain the desired identity (1.1.2). □

Corollary 1.1.4. *Suppose that u is a non-negative continuous solution of $\partial u/\partial t = \Delta\varphi(u)$ in $D'(\Omega)$, $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}$. Let $\varphi_k \in C^\infty([0,\infty)) \cap \Gamma_a$ and $\varphi_k \to \varphi$ uniformly on compact subsets of $[0,\infty)$. Then for $Q \subset\subset \Omega$, there are non-negative solutions $u_k \in C^\infty(Q)$ of $\partial u_k/\partial t = \Delta\varphi_k(u_k)$ in Q, that converge uniformly to u on compact subsets of Q.*

Proof. The corollary follows easily from Corollary 1.1.2 and Proposition 1.1.3. □

1.2 A priori L^∞-bounds for slow diffusion

In this section we shall first deal with smooth, non-negative solutions of $\partial u/\partial t = \Delta\varphi(u)$, where φ belongs to $C^\infty([0,\infty)) \cap \mathcal{S}_a$. Our aim is to establish pointwise estimates of u in terms of averages. The proof given here [45] is based on a variant of the Moser [109] iteration technique.

We shall use the following notation:

$$\begin{aligned} Q^* &= \left\{(x,t) \in \mathbb{R}^{n+1} : |x| < 2,\ -4 < t < 0\right\}, \\ Q &= \left\{(x,t) \in Q^* : |x| < 1,\ -1 < t < 0\right\}, \\ R &= B_\rho \times (-\tau, 0], \\ S &= B_r \times (-T, 0] \end{aligned}$$

with $0 < r < \rho$, $0 < T < \tau$.

Theorem 1.2.1. *Let u be a smooth non-negative solution of the equation $\partial u/\partial t = \Delta\varphi(u)$ in Q^*, with $\varphi \in \mathcal{S}_a$. Then*

$$\|u\|_{L^\infty(Q)} \le C \left\{1 + \iint_{Q^*} u^p \, dx dt\right\}^{\theta/p} \tag{1.2.1}$$

where C, p, θ are positive constants which depend only on a and n.

To prove Theorem 1.2.1 we need first some preliminary estimates:

Lemma 1.2.2. *Let v be a smooth non-negative solution of $\partial v/\partial t \le \Delta\varphi(v)$ in R. Then for $\alpha \ge \varepsilon > 0$ there is a constant $C_\varepsilon = C_\varepsilon(a,n)$ such that*

$$\begin{aligned} &\int_{B_r} \alpha\, \Phi_\alpha(v(x,0)) dx + \iint_S \left|\nabla\varphi^\beta(v)\right|^2 dx dt \\ &\qquad \le C_\varepsilon \left\{(\rho - r)^{-2} + (\tau - T)^{-1}\right\} \iint_R \left[\varphi^{2\beta}(v) + \alpha\, \Phi_\alpha(v)\right] dx dt \end{aligned} \tag{1.2.2}$$

where

$$\Phi_\alpha(v) = \int_0^v [\varphi(s)]^\alpha ds \tag{1.2.3}$$

and

$$\beta = \frac{\alpha+1}{2}.$$

Proof. We begin by choosing $\psi \in C^\infty(\mathbb{R}^{n+1})$ such that

(i) $0 \le \psi \le 1$,

(ii) $\psi \equiv 1$ on S and $\psi \equiv 0$ outside R,

(iii) $|\nabla\psi(x,t)| \le C\,(\rho - r)^{-1}$ and $|\partial_t \psi(x,t)| \le C\,(\tau - T)^{-1}$ in $R \setminus S$.

By hypothesis, if η is a Lipschitz non-negative function supported in R, then

$$\iint_R [\,\eta\,\partial_t v + \nabla\eta \cdot \nabla\varphi(v)\,]dxdt \le 0.$$

Taking $\eta = \psi^2 \varphi^\alpha(v)$ we obtain

$$\iint_R \left[\varphi^\alpha(v)\partial_t v + \alpha\varphi^{\alpha-1}(v)\ |\nabla\varphi(v)|^2\right]\psi^2\,dxdt \le -2\iint_R \varphi^\alpha(v)\,\nabla\varphi(v)\cdot\nabla\psi\ \psi\,dxdt.$$

Hence, for any $\delta > 0$,

$$\begin{aligned}\iint_R &\left[\,\varphi^\alpha(v)\partial_t v + \alpha\beta^{-2}\left|\nabla\varphi^\beta(v)\right|^2\,\right]\psi^2 dxdt \\ &\le -2\beta^{-1}\iint_R \varphi^\beta(v)\nabla\varphi^\beta(v)\,\nabla\psi\ \psi\,dxdt \\ &\le \delta\beta^{-1}\iint_R \left|\nabla\varphi^\beta(v)\right|^2\psi^2\,dxdt + \delta^{-1}\beta^{-1}\iint_R \varphi^{2\beta}(v)\ |\nabla\psi|^2\ dxdt.\end{aligned}$$

Choosing $\delta = \alpha/2\beta$ and combining terms we find that

$$\iint_R \varphi^\alpha(v)\,\partial_t v\,\psi^2\,dxdt + \frac{\alpha\beta^{-2}}{2}\iint_R \left|\nabla\varphi^\beta(v)\right|^2\psi^2\,dxdt \le 2\alpha^{-1}\iint_R \varphi^{2\beta}(v)\ |\nabla\psi|^2\ dxdt.$$

Let Φ_a be as in (1.2.3). Integrating by parts in t and using that

$$\partial_t\Phi_\alpha(v) = \partial_t \int_0^v [\varphi(s)]^\alpha ds = \varphi^\alpha(v)\,\partial_t v$$

we obtain that

$$\int_{B_r} \alpha\ \Phi_\alpha(v(x,0))\,\psi^2(x,0)dx + \frac{\alpha^2\beta^{-2}}{2}\iint_R \left|\nabla\varphi^\beta(v)\right|^2\psi^2\,dxdt \le 2\iint_R \varphi^{2\beta}(v)\ |\nabla\psi|^2\,dxdt + 2\iint_R \alpha\ \Phi_\alpha(v)\psi\ \ |\partial_t\psi|\ dxdt.$$

From the hypotheses on ψ and using that $\beta = \frac{\alpha+1}{2}$ we obtain (1.2.2), when $\alpha \geq \varepsilon > 0$. □

Corollary 1.2.3. *Let v be as in Lemma 1.2.2 and $\alpha \geq \varepsilon > 0$. Then*

$$\sup_{t\in[-T,0]} \left\{ \int_{B_r} \alpha\, \Phi_\alpha\,(v(x,t))\,dx \right\} + \iint_S \left|\nabla \varphi^\beta(v)\right|^2 dxdt \\ \leq C_\varepsilon \left\{(\rho - r)^{-2} + (\tau - T)^{-1}\right\} \iint_R \left[\varphi^{2\beta}(v) + \alpha\, \Phi_\alpha(v)\right] dxdt \tag{1.2.4}$$

with Φ_a given by (1.2.3).

Proof. Pick $t_0 \in [0, T)$ such that

$$\int_{B_r} \Phi_\alpha\,(v(x, -t_0))\,dx \geq \frac{1}{2} \sup_{t\in[-T,0]} \int_{B_r} \Phi_\alpha\,(v(x,t))\,dx.$$

Using Lemma 1.2.2 with R replaced by $R' = B_\rho \times (-\tau, -t_0]$ and S replaced by $S' = B_r \times (-T, -t_0]$ we obtain (1.2.4). □

In the proof of Theorem 1.2.1, we shall use the following variant of Sobolev's inequality.

Lemma 1.2.4. *Let w be a non-negative smooth function defined in R. Let $q^* = q/(q-1)$, $q = n/2$ for $n \geq 3$, and $q = 2$ for $n = 1, 2$. Then for $k \in (1, q^*)$, we have*

$$\rho^{-n}\,\tau^{-1} \iint_R w^{2k} dxdt \leq C \left\{ \rho^{-n}\,\tau^{-1} \iint_R (w^2 + \rho^2\,|\nabla w|^2)\,dxdt \right. \\ \left. \cdot \sup_{t\in(-\tau,0)} \left(\rho^{-n} \int_{B_\rho} (w(x,t))^{2(k-1)q}\,dx\right)^{1/q} \right\} \tag{1.2.5}$$

where C depends only on n.

Proof. Since the estimate (1.2.5) is scaling invariant we can assume that $\rho = \tau = 1$. Using Hölder and Sobolev inequalities we have that

$$\begin{aligned}\int_{B_1} w^{2k}(x,t)dx &= \int_{B_1} w^2(x,t) w^{2(k-1)}(x,t)\,dx \\ &\leq \left(\int_{B_1} w^{2q^*}(x,t)dx\right)^{1/q^*} \left(\int_{B_1} w^{2(k-1)q}(x,t)dx\right)^{1/q} \\ &\leq C \left(\int_{B_1} (w^2 + |\nabla w|^2)dx\right) \left(\sup_{t\in(-1,0)} \int_{B_1} w^{2(k-1)q}(x,t)dx\right)^{1/q}.\end{aligned}$$

By integrating in time we obtain (1.2.5). □

Proposition 1.2.5. *For any $q \geq 1$ there exists $\alpha_0 \geq 1$ such that if $\alpha \geq \alpha_0$ and $u \geq 1$ then*

$$\varphi^{2\beta(k-1)q}(u) \leq M_\alpha \, \Phi_\alpha(u) \tag{1.2.6}$$

where

$$k = k(\alpha) = \frac{1 + 1/q}{1 + 1/\alpha}, \quad \beta = \frac{\alpha + 1}{2}$$

and

$$M_\alpha = \max\left(\frac{\alpha + a}{a}, \frac{1}{a}\right).$$

Proof. Since $\varphi \in \mathcal{S}_a$

$$\int_x^1 \frac{\varphi'(u)}{\varphi(u)} du \leq \frac{1}{a} \int_x^1 \frac{du}{u} \quad \text{for any } x \in (0, 1]$$

Hence, $\varphi(x) \geq x^{1/a}$ for $x \in (0, 1]$, and

$$\Phi_\alpha(1) = \int_0^1 \varphi^\alpha(s) ds \geq \int_0^1 s^{\alpha/a} ds = \frac{a}{\alpha + a}.$$

Therefore, taking M_α sufficiently large (1.2.6) holds at $u = 1$.

On the other hand, we have

$$\begin{aligned} u \frac{d}{du}[\, M_\alpha \, \Phi_\alpha(u) - (\varphi(u))^{2\beta(k-1)q} \,] & \\ &\geq \alpha \, M_\alpha u \, \varphi^\alpha(u) - 2a^{-1}\beta\,(k-1)q\,(\varphi(u))^{2\beta(k-1)q} \\ &\geq \varphi^\alpha(u) \left[\alpha M_\alpha u - 2a^{-1}\beta\,(k-1)q\right] \geq 0 \end{aligned}$$

for $u \geq 1$ and $M_\alpha \geq a^{-1}$. Observe that $k = \frac{1+1/q}{1+1/\alpha}$ (hence in particular $k > 1$ if $\alpha \geq \alpha_0 > q$) and $2\beta = \alpha + 1$, thus

$$0 < 2\beta(k-1)q = (\alpha + 1)\frac{(\alpha - q)}{(\alpha + 1)} = \alpha - q < \alpha. \qquad \square$$

Proof of Theorem 1.2.1. To simplify the notation we set

$$v = \max\{u, 1\} \quad \text{and} \quad F = (\rho - r)^{-2} + (\tau - T)^{-1}.$$

Observe that because of growth condition (1.0.3), $\varphi(v) \geq v$, therefore

$$\Phi_\alpha(v) \leq v\,\varphi^\alpha(v) \leq \varphi^{\alpha+1}(v). \tag{1.2.7}$$

Assume now that $Q \subseteq S \subset R \subset Q^*$. By Lemma 1.2.4 with $w = (\varphi(v))^\beta$ we find

$$\begin{aligned} \iint_S \varphi^{2\beta k}(v) dx dt \leq C \Bigg\{ \iint_S \varphi^{2\beta}(v) + \left|\nabla \varphi^\beta(v)\right|^2 \, dx dt & \\ \cdot \sup_{t \in (-T, 0)} \left(\int_{B_r} \varphi^{2\beta(k-1)q}(v) dx\right)^{1/q} \Bigg\}. \end{aligned} \tag{1.2.8}$$

By Proposition 1.2.5 we have $\varphi^{2\beta(k-1)q}(v) \le \alpha\,\Phi_a(v)$, for $\alpha \ge \alpha_0 \ge 1$. Hence by 1.2.3 and the above remarks we obtain the estimate

$$\begin{aligned}
&\iint_S (\varphi(v))^{2\beta k}\,dxdt \\
&\quad\le C\,F\left[\iint_R \Big((\varphi(v))^{2\beta} + \alpha\,\Phi_\alpha(v)\Big)\,dxdt\right]\cdot \sup_{t\in[-T,0]}\left(\int_{B_r} \alpha\,\Phi_\alpha(v)dx\right)^{1/q} \\
&\quad\le C\,F^{1+1/q}\,\alpha\left(\iint_R (\varphi(v))^{2\beta}\,dxdt\right)^{1+1/q}
\end{aligned} \tag{1.2.9}$$

where $C = C(n,\alpha_0)$, for $\alpha \ge \alpha_0$.

Define next the sequence $\alpha_0, \alpha_1, \ldots$ and $\beta_0, \beta_1, \ldots$ inductively by letting α_0 be as in Proposition 1.2.5, and

$$\beta_\nu = \frac{\alpha_\nu + 1}{2}, \qquad \beta_{\nu+1} = k(\alpha_\nu)\beta_\nu \quad \text{and} \quad k(\alpha) = \frac{1+1/q}{1+1/\alpha} \tag{1.2.10}$$

with $q = n/2$, when $n \ge 3$ and $q = 2$ otherwise. Also, define

$$r_\nu = \frac{2(1+\nu)}{1+2\nu} \quad \text{and} \quad R_\nu = \{(x,t) : |x| < r_\nu,\ -r_\nu^2 < t < 0\}$$

and

$$M_\nu = \left(\iint_{R_\nu} \varphi^{2\beta_\nu}(v)\,dxdt\right)^{1/2\beta_\nu}.$$

It then follows from (1.2.9) that

$$M_{\nu+1}^{2\beta_{\nu+1}} \le C\,F_\nu^{1+1/q}\,\alpha_\nu\,M_\nu^{2\beta_\nu\,(1+1/q)}.$$

Using (1.2.10) and the estimate $F_\nu \le C\,r_\nu^2 \le C\,\nu^4$, we conclude that

$$M_{\nu+1} \le [C\,\nu^8\,\alpha_\nu]^{1/(\alpha_{\nu+1}+1)}\,M_\nu^{\theta_\nu} \tag{1.2.11}$$

with $\theta_\nu = (1+1/q)/k(\alpha_\nu)$.

Since $\lim_{\alpha\to\infty} k(\alpha) = 1 + 1/q$, it follows that $E^\nu \le \alpha_\nu \le (E^*)^\nu$ for some numbers $1 < E < E^* < \infty$. Thus from (1.2.11) we obtain that

$$M_{\nu+1} = e^{\gamma_\nu} M_\nu^{\theta_\nu},$$

where $0 \le \gamma_\nu \le C\,(\nu+1)E^{-\nu}$. Observe that $1 < \theta_\nu < 1 + C\,E^{-\nu}$, and so it easily follows that $\lim_{\nu\to\infty} M_\nu \le C\,M_0^{\theta_0}$ which proves Theorem 1.2.1. $\square$

We shall now give a rescaled version of Theorem 1.2.1. Let $Q_r = B_r \times (-r^2, 0]$.

Corollary 1.2.6. *Let u be a smooth non-negative solution of the equation $\partial u/\partial t = \Delta\varphi(u)$ in Q_ρ, with $1/2 < \rho < 2$. Then, for any $1/2 < r < \rho < 2$,*

$$\|u\|_{L^\infty(Q_r)} \le C\left\{1 + \frac{1}{(\rho - r)^N}\iint_{Q_\rho} u^p\,dx\,dt\right\}^{\theta/p} \tag{1.2.12}$$

where C, p, θ, N are positive constants which depend only on a and n.

We shall now estimate the maximum of a solution in terms of spatial averages.

Lemma 1.2.7. *Let u be a smooth non-negative solution of the equation $\partial u/\partial t = \Delta\varphi(u)$, in Q^*, with $\varphi \in \mathcal{S}_a$. Then*

$$\|u\|_{L^\infty(Q)} \le C\left\{1 + \sup_{-4<t<0}\int_{|x|<2} u(x,t)\,dx\right\}^\sigma \tag{1.2.13}$$

where the constants C, σ depend only on n and a.

Proof. Let $S = B_r \times (-T, 0]$, $R = B_\rho \times (-\tau, 0]$ satisfy $Q \subset S \subset R \subset Q^*$ and set $v = \max\{u, 1\}$. From (1.2.4) and (1.2.5) for $k \in (1, q^*)$ we have that

$$\iint_S \varphi^{2\beta k}(v)\,dx\,dt$$
$$\le C(\alpha)\,F\left(\iint_R \varphi^{2\beta}(v)\,dx\,dt\right)\cdot \sup_{t\in(-\tau,0]}\left[\int_{B_\rho}\Big(\varphi(v)\Big)^{2\beta(k-1)q}dx\right]^{1/q}$$

As before, $F = (\rho - r)^{-2} + (\tau - T)^{-1}$, $\beta = (\alpha + 1)/2$.

Choose α sufficiently large so that $v^p \le \varphi(v)^{2\beta k}$ for every $k > 1$ and $v \ge 1$, with p as in Theorem 1.2.1. We shall bound $\iint_{Q'} \varphi(v)^{2\beta k}\,dx\,dt$ in terms of spatial averages, for $Q \subset Q' \subset Q^*$. This combined with Theorem 1.2.1 will imply (1.2.13).

To this end observe first that since $\varphi(v) \le v^M$, for some $M > 1$ and all $v \ge 1$, if we pick $k \in (1, q^*)$ so close to one such that

$$2\beta(k-1)qM < 1$$

it follows that

$$\iint_S \varphi^{2\beta k}(v)\,dx\,dt \le C\,F\,I^{1/q}\iint_R \varphi^{2\beta}(v)\,dx\,dt \tag{1.2.14}$$

where $I = \sup_{t\in[-4,0]}\int_{|x|<2} v(x,t)\,dx$.

For $r \in [1/2, 2]$ and $s > 0$ we define

$$L(r) = B_r \times [-r^2, 0]$$

and

$$m(r,s) = \iint_{L(r)} w^s\,dx\,dt$$

with $w = (\varphi(v))^{2\beta}$. From (1.2.14) it follows that for $1/2 < r < \rho < 2$

$$m(r,k) \le C\,(\rho - r)^{-2}\, I^{1/q}\, m(\rho, 1). \tag{1.2.15}$$

Using Hölder's inequality we have

$$m(\rho, 1) \le m(\rho, k)^{\theta/k} m(\rho, s)^{(1-\theta)/s}$$

for any $s \in (0,1)$ with $\theta = \frac{(1-s)k}{k-s} \in (0,1)$. For $\gamma > 1$ and $r \in [1/2, 1]$, (1.2.15) shows that

$$\begin{aligned}\log m(2r^\gamma, k) \le \frac{1}{q}\log I + \log C + \log(2(r - r^\gamma))^{-2}\\ + \frac{\theta}{k}\log m(2r,k) + \frac{1-\theta}{s}\log m(2,s)\end{aligned}$$

since $m(2r,s) \le m(2,s)$. Integrating in r from $3/4$ to 1 we find after a change of variable that

$$\begin{aligned}\gamma^{-1}\int_{3/4}^{1} \log m(2r,k)\frac{dr}{r} \le C_1 \,\log I + C_2\, \log m(2,s) + C_3\\ + \frac{\theta}{k}\int_{3/4}^{1}\log m(2r,k)\frac{dr}{r}.\end{aligned} \tag{1.2.16}$$

Now we choose s so small that $(\varphi(v))^{2\beta s} \le v$ for all $v \ge 1$, and γ so close to 1 such that $\gamma^{-1} > \frac{\theta}{k}$.

If $m(3/2, k) \le 1$, then Theorem 1.2.1 shows that $\|u\|_{L^\infty(Q)} \le C$ and the proof is completed.

Hence, we may assume $m(3/2,k) > 1$. Thus $\log m(2r,k) > 0$ for $r \in [3/4, 1]$, and from (1.2.16) we obtain

$$\left(\gamma^{-1} - \frac{\theta}{k}\right)\int_{3/4}^{1}\log m(3/2,k)\frac{dr}{r} \le C_1\, \log I + C_2 \log m(2,s) + C_3$$

which yields

$$\iint_{L(3/2)} \varphi^{2\beta k}(v)\,dxdt \le C\, I^{\sigma_1}\left(\iint_Q \varphi^{2\beta s}(v)\,dxdt\right)^{\sigma_2}$$

where C, σ_1, σ_2 depend only on n, s, and a. By our choice of s it follows that

$$\iint_{L(3/2)} \varphi^{2\beta k}(v)\,dxdt \le C\, I^{\sigma}$$

which together with Theorem 1.2.1 completes the proof of the lemma. □

Our final result is an improvement of Lemma 1.2.7 which shows that if the spatial averages of the solution are small so is its L^∞-norm.

Theorem 1.2.8. *Suppose that u is a smooth, non-negative solution of the equation $\partial u/\partial t = \Delta\varphi(u)$ in Q^*, with $\varphi \in \mathcal{S}_a$. Then there exist constants $C, \gamma, \sigma > 0$ such that*

$$\|u\|_{L^\infty(Q)} \leq C\,\{I^\gamma + I^\sigma\} \tag{1.2.17}$$

where $I = \sup_{-4<t<0} \int_{|x|<2} u(x,t)dx$.

Proof. If $I \geq \varepsilon_0 > 0$ then from Lemma 1.2.7 we obtain (1.2.17) with a constant C depending on ε_0. If $0 \leq I \leq \varepsilon_0 < 1$ it follows from (1.2.13) that $0 \leq u < C_0 = C_0(r_0)$ in $Q_{r_0} = \{(x,t) : |x| < r_0,\ -r_0^2 < t < 0\}$, for $r_0 \in (1,2)$.

Claim. There exists $\varepsilon_0 \in (0,1)$ such that if $I \leq \varepsilon_0 = \varepsilon_0(r_0, r_1)$, then $0 \leq u \leq 1$ in Q_{r_1}, with $1 < r_1 < r_0 < 2$.

The proof of this claim readily follows by combining the equicontinuity result H.1 stated in the Introduction and the assumption on the smallness of the spatial integrals.

Thus we have reduced the problem to the following: *If $0 \leq u \leq 1$ in Q^*, then $\|u\|_{L^\infty(Q)} \leq C\,I^\sigma$.*

To prove the last statement we begin by combining Corollary 1.2.3 and Lemma 1.2.4 with $w = (\varphi(u))^\beta$ and $\beta = (\alpha+1)/2$, we obtain

$$\begin{aligned}\iint_S \varphi^{2\beta k}(u)dxdt \leq C\,F\left\{\iint_R \varphi^{2\beta}(u) + \alpha\Phi_\alpha(u)\,dxdt\right\}& \\ \cdot \sup_{[-T,0]}\left(\int_{B_r} (\varphi(u))^{2\beta(k-1)q}\ dx\right)^{1/q}&\end{aligned} \tag{1.2.18}$$

with $k > 1$ and $S, R, F, \beta, \Phi_\alpha$ as in the proof of Theorem 1.2.1.

Observe that there exists $\eta = \eta(a) > 0$ such that $u \geq \varphi^\eta(u)$, for $u \in (0,1)$. For $\alpha > 0$, define

$$k = k(\alpha) = 1 + \frac{\alpha+\eta}{q\,(\alpha+1)}$$

where $q = n/2$ for $n \geq 3$, and $q = 2$ for $n = 1, 2$. Observe that

$$\lim_{\alpha\to\infty} k(\alpha) = 1 + \frac{1}{q} \in (1, q^*), \quad q^* = \frac{q}{q-1}.$$

We choose $\alpha_0 > 0$ such that $1 < k < q^*$ for $\alpha \geq \alpha_0$. From our choice of k and η, there exists $M > 0$ such that for $u \in [0,1]$

$$\begin{aligned} u\,\frac{d}{du}&\left\{\alpha M\Phi_\alpha(u) - (\varphi(u))^{2\beta(k-1)q}\right\} \\ &\geq \alpha\,M\,u\,\varphi^\alpha(u) - C\,(\alpha+1)(k-1)q\,(\varphi(u))^{\alpha+\eta} \geq 0\end{aligned}$$

since $u \geq \varphi(u)^\eta$. Since $\Phi_a(0) = \varphi(0) = 0$, we conclude that

$$(\varphi(u))^{2\beta(k-1)q} \leq M\alpha\,\Phi_\alpha(u).$$

On the other hand for $u \in [0, 1]$

$$\Phi_\alpha(u) \le u\,\varphi^\alpha(u) \le \varphi^\alpha(u).$$

Thus from Lemma 1.2.2 and (1.2.18) we find that

$$\iint_S \varphi^{2\beta k}(u)\,dx dt \le C\,(\alpha F)^{1+1/q} \left(\iint_R \varphi^\alpha(u)\,dx dt\right)^{1+1/q} \tag{1.2.19}$$

because also $\varphi^{2\beta}(u) = \varphi^{\alpha+1}(u) \le \varphi^\alpha(u)$, for $u \in [0, 1]$.

Arguing in a similar manner to the proof of Theorem 1.2.1 from (1.2.9), it follows from (1.2.19) that

$$\|u\|_{L^\infty(Q)} \le C \left(\iint_{Q^*} \varphi^\alpha(u)\,dx dt\right)^{\sigma/\alpha}$$

for $\alpha \ge \alpha_0$ and $\sigma = \sigma(a, n)$. Choosing $\alpha > \alpha_0$ such that $\varphi^\alpha(u) \le u$ for $u \in [0, 1]$ we finish the proof. □

1.3 Harnack inequality for slow diffusion

In this section we shall consider non-negative solutions of the generalized porous medium equation $\partial u/\partial t = \Delta\varphi(u)$, where φ belongs to $C^\infty([0, \infty)) \cap \mathcal{S}_a$, where $\mathcal{S}_a$ is the subclass of Γ_a, corresponding to slow diffusion, defined by the growth conditions (1.0.2) and (1.0.3). We shall establish a suitable Harnack inequality which controls the size of the spatial averages in terms of the value of the solution at one point. This result will be used to prove the existence of an initial trace for non-negative solutions of the generalized porous medium equation. In the special case of the porous medium equation $\phi(u) = u^m$, $m > 1$, the Harnack estimate was first established by Aronson and Caffarelli [8]. Their proof used, among other things, the explicit formula of the solution of the porous medium equations with initial trace the Dirac measure (i.e. the Barenblatt solution described in the introduction). Ughi [122] extended this result to a class of nonlinearities that were asymptotically u^m, both at $u = 0$ and at $u = \infty$. The corresponding result for nonlinearities in the class $\mathcal{S}_a$ was established by Dahlberg and Kenig [45].

For simplicity of the exposition we shall first consider the case of the porous medium equation and at the end we shall explain how one can extend the result to the general class of nonlinearities $\mathcal{S}_a$.

We shall consider then non-negative solutions of the porous medium equation

$$\frac{\partial u}{\partial t} = \Delta u^m, \quad m > 1 \tag{1.3.1}$$

on the strip $S_T = \mathbb{R}^n \times (0, T)$. We shall study the class $P(M)$ of all non-negative continuous weak solutions u of (1.3.1) in $S_\infty = \mathbb{R}^n \times [0, \infty)$ such that

$$\sup_{t>0} \int_{\mathbb{R}^n} u(x,t)dx \le M.$$

Our first result is concerned with pointwise estimates in the class $P(M)$.

Lemma 1.3.1. *If $u \in P(M)$ then there exists $C = C(m, n, M) > 0$ such that*

$$u(x,t) \le C \quad \textit{for } t \ge 1$$

and

$$u(x,t) \le C \left(\frac{\rho^2}{t}\right)^{1/(m-1)} \qquad \textit{for } t \in (0,1)$$

where

$$\left(\frac{\rho^2}{t}\right)^{1/(m-1)} = t^{-n/(n(m-1)+2)}.$$

Proof. The proof is based on local estimates and rescaling arguments.

If $t > 1$ consider the interval $(t-1, t)$. By hypothesis

$$\int_{|\xi - x| \le 1} u(\xi, \tau)\, d\xi \le M$$

for any $x \in \mathbb{R}$ and any $\tau \in (t-1, t)$. Therefore from Theorem 1.2.1 we can conclude that $u(x,t) \le C$ for any $(x,t) \in \mathbb{R}^n \times (1, \infty)$.

For small t, if u is a solution of the equation (1.3.1) then so is

$$v(x,t) = \frac{u(\alpha x, \beta t)}{\gamma} \quad \text{if } \gamma = \left(\frac{\alpha^2}{\beta}\right)^{1/(m-1)}.$$

Define

$$v(\xi, \tau) = \gamma^{-1} u(x + \rho\xi, \tau t), \quad \text{where } \gamma = \left(\frac{\rho^2}{t}\right)^{1/(m-1)}$$

and ρ satisfies the equation

$$\rho^n \left(\frac{\rho^2}{t}\right)^{1/(m-1)} = 1.$$

Since $\gamma^{-1} \rho^{-n} = 1$ we then have

$$\int_{\mathbb{R}^n} v(\xi,\tau)d\xi = \gamma^{-1}\rho^{-n} \int_{\mathbb{R}^n} u(y, \tau t)\, dy \le M.$$

Hence $v \in P(M)$ and $v(0,1) = \frac{u(x,t)}{\gamma} \le C$. Thus

$$u(x,t) \le C \left(\frac{\rho^2}{t}\right)^{1/(m-1)}$$

as desired. □

Next we establish the existence of trace in the class $P(M)$.

Lemma 1.3.2. *If $u \in P(M)$ then there exists a positive measure μ such that*

(i) $\int d\mu \leq M$,

(ii) $u(\cdot, t) \to d\mu$ *in* $D'(\mathbb{R}^n)$ *as* $t \downarrow 0$.

Proof. From Lemma 1.3.1 we have

$$u^{m-1}(x,t) \leq C\, t^{-\sigma}$$

for $t \in (0,1)$ with

$$\sigma = \frac{n(m-1)}{n(m-1)+2} \in (0,1).$$

Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be a test function. Then for $0 < \tau < t$

$$\left| \int_{\mathbb{R}^n} [\, u(x,t) - u(x,\tau)\,]\, \eta(x)\, dx \right| \equiv \left| \int_\tau^t \int_{\mathbb{R}^n} \Delta\eta(x)\, u^m(x,s)\, dx ds \right|$$
$$\leq C_\eta \int_\tau^t \int s^{-\sigma} u(x,s)\, dx ds \leq C_\eta\, M \int_\tau^t s^{-\sigma}\, ds \leq C_\eta\, M\, t^{1-\sigma}$$

(the equality above can be justified by the limit process proved in Section 1.1). Thus $\lim_{t\downarrow 0} \int u(x,t)\, \eta(x)\, dx$ exists and the lemma is proved. □

Corollary 1.3.3. *Let $u_1, u_2 \in P(M)$ have the same trace. Then $u_1 \equiv u_2$ in $\mathbb{R}^n \times (0,\infty)$.*

Proof. If follows immediately by combining the H.3 result stated in the Introduction and Lemma 1.3.1. □

The following compactness result will be used in this section.

Lemma 1.3.4. *Suppose that $u_k \in P(M)$ with trace μ_k for $k = 1, 2, \ldots$ such that $\mu_k \to \mu$ in $D'(\mathbb{R}^n)$. Then there exists a unique $u \in P(M)$ such that $u_k \to u$ uniformly on compact sets and u has trace μ.*

Proof. By hypothesis and Theorem 1.2.1 it follows that the sequence $\{u_k\}$ is uniformly bounded. Hence it is equicontinuous (by the result H.1 stated in the Introduction).

Let u be the uniform limit on compact sets of some subsequence $\{u_{k_j}\}$ of $\{u_k\}$. Then u is a continuous non-negative weak solution of the equation (1.3.1). Also, by Fatou's Lemma, $\int u(x,t)\, dx \leq M$ for any $t \in (0,\infty)$. Moreover since, as in the proof of Lemma 1.3.2, we have for any $\eta \in C_0^\infty(\mathbb{R}^n)$

$$\left| \int_{R^n} [u_k(x,t) - u_k(x,\tau)\,]\, \eta(x)\, dx \right| \leq C_\eta\, M\, t^{1-\sigma}$$

independently of k, by taking the limit when k_j tends to infinity and t tends to zero we prove that u has trace μ. Hence, by Corollary 1.3.3 any subsequence of $\{u_k\}$ will converge to the same limit and the proof is complete. □

The following result is concerned with the existence of the fundamental solution. In the case of the porous medium equation (1.3.1) the following result follows immediately from the explicit form of the fundamental solution (the Barenblatt–Prattle self-similar solution described in the Introduction). However, we give here an independent proof of its existence and bound from below which also valid for more general nonlinearities $\varphi(u)$, with $\varphi \in \mathcal{S}_a$.

Lemma 1.3.5. *There exists $Q \in P(1)$ with initial trace the Dirac mass at the origin δ. Moreover, there exists $T = T(m, n) > 0$ such that*

$$Q(0, T) > 1/2. \tag{1.3.2}$$

Proof. Let $\{f_j\} \subseteq C_0^\infty(\mathbb{R}^n)$ be a sequence of radially decreasing functions such that $\int f_j \equiv 1$, and $f_j \to \delta$ in $D'(\mathbb{R}^n)$.

Let us denote by u_j the solution of the porous medium equation in $P(1)$ with data f_j which is radially decreasing. The existence of u_j follows from the result stated in Theorem H.2 in the Introduction. By Lemma 1.3.4 we conclude that there exists $Q \in P(1)$ with initial trace δ.

Now we pick $\eta \in C_0^\infty(\mathbb{R}^n)$ such that $\int \eta(x)dx = 1$ and $\eta(0) = 1$. Then $u_j(0, T) \geq \int u_j(x, T)\, \eta(x)dx$ and by the argument in Lemma 1.3.4

$$\left| \int_{\mathbb{R}^n} u_j(x, T)\eta(x)\, dx - 1 \right| \leq C\, T^{1-\sigma}$$

which completes the proof. □

Using the estimate (1.3.2) we shall give a preliminary version of our Harnack inequality.

Theorem 1.3.6. *Let u be a non-negative continuous weak solution of equation (1.3.1) in $S_1 = \mathbb{R}^n \times [0, 1]$. Then there exists $C = C(m, n) > 0$ such that*

$$\int_{|x|<1} u(x, 0)dx \leq C\, H_m\,(u(0, 1)) \tag{1.3.3}$$

where

$$H_m(s) = \begin{cases} 1 & \text{for } s \in (0, 1), \\ s^{1+\frac{(m-1)\,n}{2}} & \text{for } s \geq 1. \end{cases}$$

Proof. We divide the proof in two cases.

Case 1. Assume that the support of $u(\cdot, 0)$ is contained in the ball $B_1 = \{x : |x| \leq 1\}$ and that $u \in P(M)$ for some $M > 0$.

We shall use a contradiction argument. If (1.3.3) does not hold, then for each $k = 1, 2, \ldots$ there exists $u_k \in P(M_k)$ with $\operatorname{supp} u_k(\cdot, 0) \subseteq B_1$ such that

$$I_k = \int_{\mathbb{R}^n} u_k(x, 0)dx \geq k\, H_m\,(u_k(0, 1))\,.$$

Observe that $I_k \geq k$ since $H_m(\cdot) \geq 1$.

Define α_k such that

$$\alpha_k^n \cdot (\alpha_k^2)^{1/(m-1)} = I_k, \quad \gamma_k = \alpha_k^{2/(m-1)},$$

and $v_k(x,t) = \frac{u_k(\alpha_k x, t)}{\gamma_k}$ (rescaled solution).

Thus $\alpha_k \to \infty$ as $k \to \infty$, v_k is a solution of (1.3.1) with $\int v_k(x,0)dx = 1$ and $v_k \in P(1)$ by the result H.2 stated in the Introduction and with supp $v_k(\cdot,0) \subseteq B_{1/\alpha_k}(0)$. Therefore $v_k(x,0) \to \delta$ in $D'(\mathbb{R}^n)$ and, by Lemma 1.3.4, $v_k \to Q$ (fundamental solution) uniformly on compact sets. Thus

$$v_k(0,1) \to Q(0,1) > 0. \tag{1.3.4}$$

On the other hand since $v_k(0,1) = \frac{u_k(0,1)}{\gamma_k}$ then $u_k(0,1) \to +\infty$. Finally

$$I_k \geq k\, H_m\,(u_k(0,1)) = k\; (u_k(0,1))^{1+\frac{n(m-1)}{2}}$$

hence

$$I_k = \alpha_k^{n+\frac{2}{m-1}} \geq k\,(u_k(0,1))^{\frac{2+n(m-1)}{2}}.$$

Thus $\alpha_k^{2/(m-1)} \geq \sqrt[q]{k}\, u_k(0,1)$, with $q = (2+n(m-1))/2$. In other words

$$k^{-1/q} \geq \frac{u_k(0,1)}{\alpha_k^{2/(m-1)}} = v_k(0,1).$$

Therefore $v_k(0,1) \to 0$ when $k \to \infty$, which contradicts (1.3.4), hence finishing the proof.

Case 2. *General case.*

Denote by h a continuous function in $\mathbb{R}^n$ such that $0 \leq h \leq 1$ with support contained in $B_1 = \{x : |x| \leq 1\}$, and consider the BVP

$$\left.\begin{array}{ll} \partial w_R/\partial t = \Delta w_R^m & \text{in } C_R = \{(x,t) :\ |x| < R,\ 0 < t < R\}, \\ w_R(x,0) = h(x)u(x,0) & \text{for } |x| < R, \\ w_R(x,t) = 0 & \text{for } |x| = R \text{ and } t \in (0,R). \end{array}\right\} \tag{1.3.5}$$

The existence of a solution w_R of (1.3.5) will be proved in a following section. The uniqueness follows by Theorem 1.1.1. In fact, for $R < \rho$ we have that $w_R \leq w_\rho \leq C$ in C_R.

Let v be the solution of (1.3.1) in $\mathbb{R}^n \times [0,\infty)$ with data $h(x)u(x,0)$ (this existence result follows by H.2). Then by Theorem 1.1.1

$$w_R \leq w_\rho \leq v \quad \text{in } C_R \text{ with } v \in P(M).$$

By the equicontinuity result H.1 there exists sequence $\{\rho_k\}$ such that $w_{\rho_k} \to w$ uniformly on compact sets with $w \in P(M)$. From Case 1 we have that

$$\int_{|x|\leq 1} h(x)\, u(x,0)\, dx \leq C\, H_m\,(w(0,1)) \leq C\, H_m\,(u(0,1))$$

which completes the proof. □

Now we shall rescale the estimate (1.3.3).

Theorem 1.3.7. *Let u be a non-negative continuous weak solution of the porous medium equation* (1.3.1) *in* $S_T = \mathbb{R}^n \times [0,T]$. *Then for* $R > T^{1/2}$

$$\int_{|x|<R} u(x,0)dx \leq C\left\{R^n \left(\frac{R^2}{T}\right)^{1/(m-1)} + T^{n/2}\, H_m\,(u(0,T))\right\}. \tag{1.3.6}$$

Proof. Define $\gamma = \left(\frac{R^2}{T}\right)^{1/(m-1)}$ and $v(x,t) = \frac{u(Rx,Tt)}{\gamma}$. Then v is also a solution of the porous medium equation and by (1.3.3)

$$\begin{aligned}\int_{|x|<1} v(x,0)dx &= \left(R^n \left(\frac{R^2}{T}\right)^{1/(m-1)}\right)^{-1} \int_{|x|<R} u(x,0)dx \\ &\leq C\, H_m\,(v(0,1)) \\ &\leq C\left\{1 + \left[R^n \left(\frac{R^2}{T}\right)^{1/(m-1)}\right]^{-1} T^{n/2}\, H_m\,(u(0,T))\right\}.\end{aligned}$$

□

We shall finish this section by explaining how to extend the above to the equation $\partial u/\partial t = \Delta\varphi(u)$ with $\varphi \in \mathcal{S}_a$. For the detailed proof of the results we refer to [45].

For $\varphi \in \mathcal{S}_a$, $\frac{\varphi(u)}{u}$ is monotonically increasing on $[1,\infty)$. Denote by Λ its inverse.

(i) In Lemma 1.3.1, $\left(\frac{\rho^2}{t}\right)^{1/m-1}$ must be replaced by $\Lambda\left(\frac{\rho^2}{t}\right)$.

(ii) In Theorems 1.3.6 and 1.3.7, in the definition of $H_m(s)$ for $s \geq 1$, $s^{1+\frac{(m-1)n}{2}}$ must be replaced by $s\left(\frac{\varphi(s)}{s}\right)^{n/2}$.

(iii) In Theorem 1.3.7, $\left(\frac{R^2}{T}\right)^{1/(m-1)}$ must be replaced by $\Lambda\left(\frac{R^2}{T}\right)$.

1.4 Local L^∞-bounds for fast diffusion

In this section we shall establish local L^∞-bounds for smooth, non-negative solutions of $\partial u/\partial t = \Delta\varphi(u)$, where φ belongs to $C^\infty([0,\infty)) \cap \mathcal{F}_a$. The subclass $\mathcal{F}_a$ of Γ_a, corresponding to fast diffusion, is defined by the conditions

(i) $\varphi\colon [0,\infty) \to \mathbb{R}$ is a continuous non-negative function with $\varphi(0) = 0$,

(ii) there exist $a \in (0,1)$ such that for all $u > 0$

$$a \le \frac{u\,\varphi'(u)}{\varphi(u)} \le a^{-1},$$

(iii) there exist $a \in (0,1)$ and $u_0 > 0$ such that for $u \ge u_0$

$$a + \frac{n-2}{n} \le \frac{u\,\varphi'(u)}{\varphi(u)} \le 1 - a, \tag{1.4.1}$$

(iv) $u_0 = 1$, and $\varphi(1) = 1$.

Here the lower bound in (1.4.1) enters (and of course, it necessarily enters (see [25] and [81])).

The main result in this section is the following estimate [47] (and [81] for the pure power case).

Theorem 1.4.1. *Let u be a continuous, non-negative weak solution of $\partial u/\partial t = \Delta\varphi(u)$ in $Q^* = \{(x,t) : |x| < 2, -4 < t < 0\}$, with $\varphi \in \mathcal{F}_a$. Let*

$$Q = \{(x,t) : |x| < 1, -1 < t < 0\}$$

and define $H_\varphi(s)$ to be 1 for $0 < s < 1$, and $s\,[\varphi(s)/s]^{n/2}$ for $s \ge 1$. Then there is a constant $C = C(a,n) > 0$ such that

$$\|H_\varphi(u)\|_{L^\infty(Q)} \le C\,\{\|u\|_{L^1(Q^*)} + 1\}. \tag{1.4.2}$$

The proof of Theorem 1.4.1 will be based on the following a-priori estimate.

Lemma 1.4.2. *Let u be a smooth, non-negative solution of the equation $\partial u/\partial t = \Delta\varphi(u)$ in $R = B_\rho \times (-\rho^2, 0]$, where $\varphi \in \mathcal{F}_a \cap C^\infty([0,\infty))$. Then,*

$$\|u\|_{L^\infty(S)} \le C \left\{1 + \frac{1}{(\rho - r)^N} \iint_R u^p\,dx\,dt\right\}^\theta \tag{1.4.3}$$

where C, p, θ, N are positive constants which depend only on a and n and $S = B_r \times (-r^2, 0]$, $1/2 < r < \rho < 2$.

The proof of Lemma 1.4.2 is similar to the proof of Corollary 1.2.6 and is left to the reader.

Proof of Theorem 1.4.1. First note that the lower bound in (1.4.1) implies that H_φ is increasing for $s > 1$. Next, let p be as in Lemma 1.4.2 (and we can also assume that $p \ge 1$). For such a fixed p, we define now $H_{p,\varphi}(s)$ to be 1 for $0 < s \le 1$, and

$s^p[\varphi(s)/s]^{n/2}$ for $s \geq 1$. Again, $H_{p,\varphi}$ is increasing for $s \geq 1$. Let $F_{p,\varphi}(s)$ be the inverse function to $H_{p,\varphi}$, defined again for $s \geq 1$. We first claim that

$$u(0,0) \leq C F_{p,\varphi}\left\{\iint_{Q^*} u^p(x,t)\,dxdt + 1\right\}. \tag{1.4.4}$$

In order to establish (1.4.4) we first note that if u is a continuous weak solution of $\partial u/\partial t = \Delta\varphi(u)$ in Ω, and for given $\alpha > 0, \beta > 0, \gamma > 0$ we define $v(x,t) = u(\alpha x, \beta t)/\gamma$, then v is a continuous weak solution of $\partial v/\partial t = \Delta\psi(v)$ in the appropriate Ω', where $\psi(s) = \beta\varphi(\gamma s)\alpha^{-2}\gamma^{-1}$. Suppose now that $\beta/\alpha^2 \geq 1$ and $\varphi \in \mathcal{F}_a$. Let $G_\varphi(\gamma) = \gamma/\varphi(\gamma)$, which by the right-hand side inequality in (1.4.1) is increasing for $\gamma \geq 1$. Let $E_\varphi(s)$ be its inverse function defined for $s \geq 1$, and choose $\gamma = E_\varphi(\beta/\alpha^2)$. Then $\psi(1) = \beta\varphi(\gamma)\alpha^{-2}\gamma^{-1} = 1$, and it is easy to see that $\psi \in \mathcal{F}_a$.

Assume now that (1.4.4) fails. We can then find $\varphi_k \in \mathcal{F}_a$, u_k continuous weak solutions such that

$$u_k(0,0) \geq k\, F_{p,\varphi_k}\left\{\iint_{Q^*} u_k^p(x,t)dx\,dt + 1\right\}.$$

First note that $F_{p,\varphi_k}(1) = 1$, and that $F_{p,\varphi_k}(s)$ is increasing for $s \geq 1$, so that $u_k(0,0) \geq k$. Because of (1.4.3) this forces that

$$I_k = \iint_{Q^*} u_k^p(x,t)dx\,dt \to +\infty \quad \text{as } k \to \infty.$$

For α_k small (to be chosen momentarily), let

$$v_k(x,t) = \frac{u_k(\alpha_k x, t)}{\gamma_k}, \qquad \gamma_k = E_{\varphi_k}\left(\frac{1}{\alpha_k^2}\right).$$

The v_k are continuous weak solutions of $\partial v/\partial t = \Delta\psi_k(v)$, $\psi_k \in \mathcal{F}_a$ in Q^*, by our previous discussion. In addition

$$\int_{|x|<2/\alpha_k}\int_{-4}^{0} v_k^p(x,t)\,dxdt = \frac{1}{[E_{\varphi_k}(1/\alpha_k^2)]^p}\,\alpha_k^{-n} I_k.$$

Choose α_k so that

$$\frac{1}{[E_{\varphi_k}(1/\alpha_k^2)]^p}\,\alpha_k^{-n} I_k = 1,$$

or equivalently

$$I_k = \frac{[E_{\varphi_k}(1/\alpha_k^2)]^p}{1/\alpha_k^n}.$$

This is possible because $E_{\varphi_k}^p(s)/s^{n/2}$ is increasing for $s \geq 1$, by the left-hand side inequality in (1.4.1), and $I_k \to +\infty$. Moreover, $\alpha_k \to 0$ as $k \to \infty$. But then

$$\begin{aligned} v_k(0,0) &= \frac{1}{E_{\varphi_k}(1/\alpha_k^2)} u_k(0,0) \\ &\geq \frac{k}{E_{\varphi_k}(1/\alpha_k^2)} F_{p,\varphi_k}\{I_k + 1\} \\ &\geq \frac{k}{E_{\varphi_k}(1/\alpha_k^2)} F_{p,\varphi_k}(I_k) \\ &= k F_{p,\varphi_k}\left\{\frac{[E_{\varphi_k}(1/\alpha_k^2)]^p}{1/\alpha_k^n}\right\} \frac{1}{E_{\varphi_k}(1/\alpha_k^2)} \\ &= k \end{aligned}$$

by the definition of F_{p,φ_k} and E_{φ_k}. However, this contradicts (1.4.3), by our choice of α_k, and thus (1.4.4) is established.

We next note that by using translation and dilation $(x,t) \to (\alpha x, \alpha^2 t)$, (1.4.4) has, as a consequence,

$$\|u\|_{L^\infty(B_{r/2}\times(-r^2/4,0])} \leq C F_{p,\varphi}\left\{\frac{1}{r^{n+2}} \iint_{B_r\times(-r^2,0)} u^p(x,t)dx\,dt + 1\right\} \tag{1.4.5}$$

for $0 < r < 2$. Finally, using (1.4.5), translations, the dilations $(x,t) \to (\alpha x, \alpha^2 t)$, and a simple covering argument, one can show

$$\|u\|_{L^\infty(S)} \leq C F_{p,\varphi}\left\{\frac{1}{(\rho - r)^{n+2}} \iint_R u^p(x,t)dx\,dt + 1\right\} \tag{1.4.6}$$

where $S = B_r \times (-r^2, 0]$, $R = B_\rho \times (-\rho^2, 0]$, $1/2 < r < \rho < 2$.

To conclude the proof of the theorem, we use an argument which originates in the work of Hardy and Littlewood (see [70]), and which was first used in the context of the porous medium equation in [45].

We shall first show that

$$\|u\|_{L^\infty(Q)} \leq C\{\|u\|_{L^1(Q^*)} + 1\}^\sigma, \tag{1.4.7}$$

where $\sigma = \sigma(a,n)$. Once (1.4.7) is established, the theorem will follow by repeating the argument that we used to establish (1.4.6), with $p = 1$, and using (1.4.7) instead of (1.4.3).

In order to establish (1.4.7) we need to point out two properties of $F_{p,\varphi}$. First, $F_{p,\varphi}(As) \leq C A^\sigma F_{p,\varphi}(s)$, for $A \geq 1, s \geq 1$. This is an easy consequence of (1.4.1). Another easy consequence of (1.4.1) is that $F_{p,\varphi}(s^{p-1}) \leq C s^{1-\varepsilon}$, for $s \geq 1$, where $\varepsilon = \varepsilon(a,n)$.

Now, for $1/2 < r < 1$, let

$$\begin{aligned} S_r &= B_{2r} \times (-4r^2, 0] \\ m(r) &= \|u\|_{L^\infty(S_r)} \\ I &= \iint_{Q^*} u(x,t)\,dx\,dt \\ J &= \max\{I, 1\}. \end{aligned}$$

We want to show that

$$m(1/2) \le \{I + 1\}^\sigma. \tag{1.4.8}$$

If there exists r, $1/2 < r < 1$ such that $m(r) \le 1$, we are done, and hence, we can assume that $m(r) > 1$ for all r, $1/2 < r < 1$. Pick now $1/2 < r < \rho < 1$. Then (1.4.6) implies that

$$\begin{aligned} m(r) &\le C\,F_{p,\varphi}\left\{\frac{1}{(r-\rho)^{n+2}} \iint_{S_\rho} u^p(x,t)\,dxdt + 1\right\} \\ &\le C\,F_{p,\varphi}\left\{\frac{4J}{(r-\rho)^{n+2}}\,m(\rho)^{p-1}\right\} \\ &\le C\left\{\frac{J}{(r-\rho)^{n+2}}\right\}^\sigma F_{p,\varphi}(m(\rho)^{p-1}) \\ &\le C\left\{\frac{J}{(r-\rho)^{n-2}}\right\}^\sigma m(\rho)^{1-\varepsilon}. \end{aligned}$$

Choose now γ, $0 < \gamma < 1$ so that $\theta = (1-\varepsilon)/\gamma < 1$, and let $\rho = r^\gamma$. Taking logarithms, we see that

$$\log m(r) \le C\log J + C\log\frac{1}{(r-r^\gamma)} + (1-\varepsilon)\log m(r^\gamma).$$

Integrating with respect to the measure dr/r, between $1/2$ and 1, we obtain

$$\begin{aligned} \int_{1/2}^{1} \log m(r)\,\frac{dr}{r} &\le C\log J + C + \theta\int_{1/2^\gamma}^{1} \log m(r)\,\frac{dr}{r} \\ &\le C\log J + C + \theta\int_{1/2}^{1} \log m(r)\,\frac{dr}{r}. \end{aligned}$$

The inequality (1.4.8) immediately follows from this, and our theorem is established. □

Remark. In the case when $\varphi(u) = u^m$, $0 < m < 1$, the technique of the proof of Theorem 1.4.1 allows one to show

$$\|u\|_{L^\infty(Q)} \le C_q\left\{\left(\iint_{Q^*} u^q\right)^{\frac{1}{q-(1-m)n/2}} + 1\right\}, \tag{1.4.9}$$

for each q such that $q-(1-m)n/2>0$. Note that $q=1$ is allowed precisely when $m>(n-2)/n$, giving another explanation of the result in [25]. Moreover, at least in the range $m>(n-2)/n$, an inequality of the form

$$\|u\|_{L^\infty(Q)} \le \left\{\iint_{Q^*} u^q + 1\right\}^\sigma$$

can only hold if $q-(1-m)n/2>0$, as can be seen by considering the Barenblatt solutions ([7], [17], [25], [81])

$$U_\alpha(x,t) = t^{-\beta}\left\{\alpha + 2m\gamma^{-1}|x|^2 t^{-2\beta/n}\right\}^{-s},$$

where $\alpha>0$, $s=\frac{1}{1-m}$, $\gamma=\frac{2}{1-m}-n$, $\beta^{-1}=m-1+\frac{2}{n}$.

By rescaling we obtain the following general L^∞- estimate from (1.4.2).

Theorem 1.4.3. *Let u be a weak solution of equation $\partial u/\partial t=\Delta\varphi(u)$ in $B_{4R}\times(0,T)$, continuous in $\bar{B}_{4R}\times[0,T]$, with $\varphi\in\mathcal{F}_a$. Suppose that $T/R^2>1$. Then*

$$\sup_{|x|<R} H_\varphi(u(x,T)) \le C\left\{\frac{1}{T^{n/2}}\int_{|x|<4R} u(x,0)dx + E_\varphi\left(\frac{T}{R^2}\right)\cdot\left(\frac{T}{R^2}\right)^{n/2}\right\}. \quad (1.4.10)$$

1.5 Equicontinuity of solutions

In this section we shall prove the equicontinuity result H.1 stated in the introduction due to P. E. Sacks [118]. More precisely we shall show that if $\{u_k\}$ is a *uniformly bounded* sequence of continuous weak solutions of the equation $\partial u/\partial t=\Delta\varphi(u)$ in a compact domain R of $\mathbb{R}^n\times(0,T)$, then $\{u_k\}$ is equicontinuous in any compact domain $S\subset\subset R$.

Consider weak solutions of the equation

$$\frac{\partial u}{\partial t} = \Delta\varphi(u) \quad \text{in } Q_T=\Omega\times(0,T). \quad (1.5.1)$$

Denoting $\beta=\varphi^{-1}$ we can write (1.5.1) as

$$\frac{\partial\beta(u)}{\partial t} = \Delta u \quad \text{in } Q_T=\Omega\times(0,T). \quad (1.5.2)$$

The equicontinuity property is a consequence of the following theorem which constitutes the basic result in this section.

Theorem 1.5.1. *Let $u\in C^\infty(Q_T)$ be a solution of (1.5.2) with $\beta\in C^\infty(\mathbb{R})$, $\beta(0)=0$, $0<\beta'<\infty$ and*

$$0\le\mu_1(\delta)\le\beta'(s)\le\hat{\mu}_1(\delta) \quad \textit{for } |s|\ge\delta>0. \quad (1.5.3)$$

Let C_1 be a bound for $\|u\|_{L^\infty(Q_T)}$ and $\|\beta(u)\|_{L^\infty(Q_T)}$. Then, for any $Q\subset\subset Q_T$, the modulus of continuity of u in Q depends only on C_1, μ_1, $\hat{\mu}_1$, n, and $\operatorname{dist}(Q;\partial_p Q_T)$.

Notation and Remarks. Without loss of generality we can assume that μ_1 and $\hat{\mu}_1$ are monotone increasing and decreasing respectively. We shall use the following notation:

$$Q_{(x_0,t_0)}(R) = \{(x,t) : |x - x_0| < R,\ t_0 - R^2 < t < t_0\},$$
$$Q(R) = Q_{(0,0)}(R) \quad \text{and} \quad Q = Q(1),$$
$$\overset{\circ}{V}_2(Q_T) = L^\infty\big((0,T) : L^2(\Omega)\big) \cap L^2\big((0,T) : H_0^1(\Omega)\big).$$

Recall the

Sobolev Embedding Theorem. For $u \in \overset{\circ}{V}_2(Q_T)$, we have

$$\|u\|_{L^{\frac{2(n+2)}{n}}(Q_T)} \le C_0 \|u\|_{\overset{\circ}{V}_2(Q_T)}. \tag{1.5.4}$$

We shall first present a simpler proof of this result, due to Bouillet, Caffarelli and Fabes [26], in the special case where the nonlinearity β satisfies, in addition to (1.5.3), the growth condition:

$$\beta'(s) \ge \eta > 0, \quad s > 0 \tag{1.5.5}$$

together with the assumption $\beta(0) = 0$. This is the case of the porous medium equation, where $\beta(u) = u^{1/m}$, $m > 1$. We shall show the following:

Theorem 1.5.2. *Let $u \in C^\infty(Q)$, $0 \le u \le 1$ be a solution of equation (1.5.2) in Q with β satisfying (1.5.5) and $\beta(0) = 0$. Then*

$$|u(x,t) - u(0,0)| \le \sigma(|x|, |t|), \tag{1.5.6}$$

where σ is a modulus of continuity depending only on $\beta(1)$, η and the dimension n.

We shall present the proof due to Bouillet, Caffarelli and Fabes. It is based upon two basic lemmas. The first uses the ideas of De Giorgi:

Lemma 1.5.3. *Under the hypotheses of Theorem 1.5.2, there exists a constant μ, depending only on $\beta(1)$, η and the dimension n, such that if*

$$|Q \cap \{u \le 1/2\}| \ge \mu, \tag{1.5.7}$$

then

$$\sup_{Q(1/2)} u \le \frac{3}{4}. \tag{1.5.8}$$

Proof. Throughout this lemma, C will denote various constants which depend only on $\beta(1)$, η and the dimension n. For $k \ge 1$, let $u_k = (u - 3/4 + 1/2^{k+1})^+$. Using energy estimates we shall show that

$$\iint_{Q(1/2)} u_k^2\, dx\, dt \to 0 \quad \text{as } k \to \infty,$$

provided that the measure of the set $Q \cap \{u \le 1/2\}$ is sufficiently close to one.

Let $\zeta(x,t)$ be a smooth test function, $0 \le \zeta \le 1$, $\zeta = 0$ near $\partial_p Q(r)$ and let $K > 0$. Multiplying (1.5.2) by $(u-K)^+\zeta^2$ and integrating over $B_r(0) \times (-r^2, \tau)$, for some $\tau \in (-r^2, 0)$, we obtain

$$\int_{-r^2}^{\tau}\int \beta'(u)(u-K)^+ u_t\,\zeta^2\,dx\,dt = -\int_{-r^2}^{\tau}\int \nabla u \nabla((u-K)^+\zeta^2)\,dxdt$$

which in combination with Hölder's inequality gives the estimate

$$\begin{aligned}\int_{-r^2}^{\tau}\int \beta'(u)(u-K)^+\,u_t\,\zeta^2\,dx\,dt + \frac{1}{2}\int_{-r^2}^{\tau}\int |\nabla(u-K)^+|^2\,\zeta^2\,dx\,dt \\ \le C\int_{-r^2}^{\tau}\int (u-K)^{+2}\,|\nabla\zeta|^2\,dx\,dt.\end{aligned} \tag{1.5.9}$$

In order to transform the first term on the left hand side of the previous estimate, we define

$$B(u) = \int_0^u \beta'(s+k)s\,ds,$$

so that

$$\frac{\partial B((u-K)^+)}{\partial t} = \beta'(u)\,(u-K)^+\,u_t.$$

Note that for $0 \le u \le 1$, we have

$$\frac{\eta}{2}u^2 \le B(u) \le u\beta(u) \le \beta(1)\,u.$$

Hence

$$\begin{aligned}\int ((u-K)^+\zeta)^2|_{t=\tau}\,dx + \int_{-r^2}^{\tau}\int |\nabla(u-K)^+\,\zeta|^2\,dx\,dt \\ \le C\int_{-r^2}^{\tau}\int (u-K)^{+2}\,|\nabla\zeta|^2 + (u-K)^+|\zeta_t|\,dx\,dt.\end{aligned}$$

Taking the supremum of the left side over $\tau \in (-r^2, 0]$ we obtain the estimate

$$\begin{aligned}\sup_{\tau\in(-r^2,0]}\int ((u-K)^+\zeta)^2|_{t=\tau}\,dx + \iint_{Q(r)} |\nabla(u-K)^{+2}\,\zeta^2|\,dx\,dt \\ \le C\iint_{Q(r)} (u-K)^{+2}\,|\nabla\zeta|^2 + (u-K)^+|\zeta_t|\,dx\,dt.\end{aligned} \tag{1.5.10}$$

On the other hand the proof of the variant of Sobolev's inequality (1.2.5) in Lemma 1.2.4 applied to $k = (n+2)/n$, $q = n/2$ gives us the estimate

$$\begin{aligned}&\iint_{Q(r)} ((u-K)^+\zeta)^{\frac{2(n+2)}{n}}\,dxdt \\ &\le C\left\{\iint_{Q(r)} |\nabla(u-K)^+\zeta|^2 + (u-K)^{+2}\zeta^2\,dxdt\right\} \\ &\cdot \sup_{\tau}\left(\int ((u-K)^+\zeta)^2|_{t=\tau}\,dx\right)^{1/q}.\end{aligned} \tag{1.5.11}$$

Combining (1.5.10) and (1.5.11) we conclude that for $p = 2(n+2)/n$ we obtain

$$\left(\iint_{Q(r)} ((u-K)^+\zeta)^p \, dxdt\right)^{2/p} \leq C \iint_{Q(r)} (u-K)^{+2}|\nabla\zeta|^2 + (u-K)^+|\zeta_t| \, dxdt. \tag{1.5.12}$$

For $k \geq 1$, set $r_k = 1/2 + 1/2^k$ and $Q_k = Q(r_k)$. Choose ζ_k to be smooth in $Q_k = Q(r_k)$, $\zeta_k = 0$ near $\partial_p Q_k$, $0 \leq \zeta_k \leq 1$, $\zeta_k = 1$ on Q_{k+1}. Moreover, ζ_k can be chosen to satisfy the bounds

$$|\nabla\zeta_k|^2 + |(\zeta_k)_t| \leq C^k$$

for some constant $C > 1$. Setting $u_k = (u-K)^+$, with $K = 3/4 - 1/2^{k+1}$ and applying (1.5.12), we obtain

$$\left(\iint_{Q_k} (u_k\zeta_k)^p \, dxdt\right)^{2/p} \leq C^k \iint_{Q_k} (u_k^2 + u_k) \, dxdt$$

from which we conclude that

$$\left(\iint_{Q_{k+1}} u_k^p \, dxdt\right)^{2/p} \leq C^k \iint_{Q_k} (u_k^2 + u_k) \, dxdt \tag{1.5.13}$$

for all k.

We shall use (1.5.13) to show that

$$\iint_{Q_{k+2}} u_{k+1} \leq C^k \left\{\iint_{Q_{k+1}} (u_k^2 + u_k)\right\}^{1+\varepsilon/2}. \tag{1.5.14}$$

We begin by observing that by Hölder's inequality

$$\iint_{Q_{k+2}} u_{k+1}^2 \leq \left(\iint_{Q_{k+2}} u_{k+1}^p\right)^{2/p} \cdot |\{u_{k+1} > 0\} \cap Q_{k+2}|^{(p-2)/p}. \tag{1.5.15}$$

Since

$$|\{u_{k+1} > 0\} \cap Q_{k+2}| \leq \left|\{u_k > 1/2^{k+2}\} \cap Q_{k+1}\right| \leq C4^k \iint_{Q_{k+1}} u_k^2 \tag{1.5.16}$$

combining (1.5.13), (1.5.15) and (1.5.16), we obtain

$$\iint_{Q_{k+2}} u_{k+1}^2 \leq C^k \left(\iint_{Q_{k+1}} (u_k^2 + u_k) \, dxdt\right)^{1+\varepsilon}, \tag{1.5.17}$$

with $\varepsilon = (p-2)/p = 2/(n+2)$. Also, since $u_k > 1/2^{k+2}$ when $u_{k+1} > 0$, we have

$$\iint_{Q_{k+2}} u_{k+1} \leq \iint_{Q_{k+2}} u_{k+1}\,(u_k 2^{k+2})$$
$$\leq C^k \left(\iint_{Q_{k+2}} u_{k+1}^2\right)^{1/2} \left(\iint_{Q_{k+1}\cap\{u_k>1/2^{k+2}\}} u_k^2\right)^{1/2}$$

and therefore from (1.5.17) we conclude (1.5.14).

Setting now $\alpha_k = \iint_{Q_{k+1}} (u_k^2 + u_k)$, (1.5.17) and (1.5.14) imply that

$$\alpha_{k+1} \leq C^k\,\alpha_k^{1+\varepsilon/2}.$$

Hence,

$$\lim_{k\to\infty} \alpha_k = 0,$$

provided that

$$\alpha_1 = \iint_Q \left(u - \frac{1}{2}\right)_+ + \left(u - \frac{1}{2}\right)_+^2 dx\,dt$$

is sufficiently small, depending only on $\beta(1)$, η and the dimension n. This readily implies that $\lim_{k\to\infty} \alpha_k = 0$ provided that the measure of the set $Q \cap \{\, u \leq 1/2\,\}$ is sufficiently close to one showing the desired result. □

The next lemma is due to Bouillet, Caffarelli and Fabes [26].

Lemma 1.5.4. *There exist positive numbers $N = N(\beta(1), \eta)$ and $\delta = \delta(\beta(1), \eta)$, depending only on $\beta(1)$, η and also on the dimension n, such that if $0 \leq u \leq 1$ is a smooth solution of (1.5.2) on the cylinder $Q_N \equiv \{|x| \leq 6\} \times (-N, 0)$, with β satisfying the assumptions of Theorem 1.5.2, then*

$$\inf_{|x|\leq 1/4} u(x,0) \geq \delta \quad \textit{or} \quad \sup_{|x|\leq 1/4} u(x,0) \leq 1-\delta. \tag{1.5.18}$$

Proof. For $j = 0, 1, \ldots, N-1$, set $Q_{j+1,j} = \{|x| < 1\} \times (-(j+1), -j)$. Since the function u solves (1.5.2), the function $v = 1 - u$ solves the equation $(\tilde{\beta}(v))_t = \Delta v$, with $\tilde{\beta}(v) = \beta(1) - \beta(1-v)$ still satisfying the hypotheses of Lemma 1.5.3. Therefore, if for some j

$$|Q_{j+1,j} \cap \{1 - u \leq 1/2\}| \geq \mu, \tag{1.5.19}$$

with μ as in Lemma 1.5.3, it follows from this lemma that

$$1 - u \leq \frac{3}{4} \quad \text{on } \left\{|x| < \tfrac{1}{2}\right\} \times \left(-j - \tfrac{1}{4}, -j\right).$$

In particular, we shall then have $u(x,-j) \geq 1/4$, for $|x| \leq 1/2$. We shall show that in this case

$$u(x,0) \geq \delta \quad \text{for } |x| \leq \tfrac{1}{4}, \tag{1.5.20}$$

for an appropriate positive constant δ, depending only on η and the dimension n. Indeed, let ϕ solve

$$-\Delta\phi = \lambda_1\phi \quad \text{in } |x| \le \frac{1}{2}, \tag{1.5.21}$$

with $\phi \equiv 0$ at $|x| = 1/2$, where $\lambda_1 > 0$ denotes the principal eigenvalue of $-\Delta$. Since, $\beta'(u) \ge \eta$, it is easy to check that for all $\alpha > 0$

$$w(x,t) = \alpha e^{-\lambda_1 t/\eta}\phi(x)$$

satisfies

$$-\frac{1}{\beta'(u)}\Delta w + w_t \le 0.$$

Hence, if α is chosen sufficiently small, depending only on dimension n and η, so that

$$u(x,-j) \ge \frac{1}{4} \ge \alpha\phi(x) = w(x,0),$$

we can deduce from the maximum principle on $\{|x| < 1/2\} \times (-j, 0)$, that

$$u(x,0) \ge w(x,j) \quad \text{for } |x| \le \frac{1}{2}.$$

Since $w(x,j) = \alpha e^{-\lambda_1 j/\eta}\phi(x)$, we conclude that (1.5.20) holds true by choosing

$$\delta < \alpha e^{-\lambda_1 N/\eta} \inf_{|x|\le 1/4} \phi. \tag{1.5.22}$$

Therefore, we may assume that

$$|Q_{j+1,j} \cap \{1-u \le 1/2\}| < \mu \quad \text{for all } j = 0, 1, \dots, N-1$$

which implies that

$$|Q_{j+1,j} \cap \{u \le 1/2\}| \ge 1-\mu = \lambda > 0 \quad \text{for all } j = 0, 1, \dots, N-1.$$

Hence, the set

$$E = \big(\{|x| \le 1\} \times (-N+1, 0)\big) \cap \left\{u \le \tfrac{1}{2}\right\},$$

has measure

$$|E| \ge (N-1)\,\lambda.$$

Let $\tilde{u}$ be a subsolution of (1.5.2) such that

$$\tilde{u} = 1/2 \quad \text{on } E \quad \text{and} \quad \tilde{u} = 1 \quad \text{on } \partial_p Q_N$$

where $\partial_p Q_N$ denotes the parabolic boundary of Q_N. In fact, $\tilde{u}$ can be chosen so that it is a solution in $Q_N \setminus E$. It follows by the maximum principle that

$$\tilde{u} \ge u \quad \text{in } Q_N.$$

Therefore, in order to show that $u(x,0) \le 1-\delta$ for $|x| \le 1/4$ it is sufficient to show the same for $\tilde{u}$. Define

$$\psi(x) = \frac{1}{N}\int_{-N}^{0} (1-\tilde{u}(x,s))\,ds.$$

Since $\psi \ge 0$ and

$$\Delta\psi \le -\frac{1}{N}\int_{-N}^{0} \big(\beta(\tilde{u}(x,s))\big)_s\,ds \le \frac{\beta(1)}{N},$$

it follows by the Harnack estimate that

$$\inf_{|x|\le 5} \psi + \frac{\beta(1)}{N} \ge \int_{|x|\le 6} \psi(x)\,dx.$$

But on the other hand

$$\int_{|x|\le 6} \psi(x)\,dx \ge \frac{1}{N}\iint_E (1-\tilde{u})\,dx \ge \frac{N-1}{N}\frac{\lambda}{2}$$

since $\tilde{u} = 1/2$ on E and the measure of E is at least $(N-1)\lambda$. Therefore we conclude that

$$\inf_{|x|\le 5} \psi + \frac{\beta(1)}{N} \ge \frac{N-1}{N}\frac{\lambda}{2}$$

and hence we can make

$$\inf_{|x|\le 5} \psi \ge \frac{\lambda}{4}$$

by choosing N sufficiently large depending on $\beta(1)$ and λ. This implies that there exists a point (x_0, t_0), with $|x_0| = 5$, $-N < t_0 < 0$ such that

$$1-\tilde{u}(x_0,t_0) \ge \frac{\lambda}{4}.$$

We shall show that

$$1-\tilde{u}(z,t_0) \ge \frac{\lambda}{4} \quad \text{for } |z| \le \frac{1}{2}. \tag{1.5.23}$$

Indeed, let Π denote the hyperplane which is the perpendicular bisector of the line segment joining z and x_0. The plane Π divides the ball $\{|x| \le 6\}$ into two parts. We denote by D_{x_0} the part which contains x_0 and D_z the domain which is obtained by reflecting D_{x_0} across the plane Π. Also, let T denote the transformation of reflection across Π. Then $1-\tilde{u}(Tx,t)$ is a solution of equation (1.5.2) in $D_z \times (-N,0)$, since $D_{x_0} \subset Q_N \setminus E$, while $1-\tilde{u}(x,t)$ is a super-solution of equation (1.5.2) in the same cylinder. Moreover

$$1-\tilde{u}(Tx,t) \le 1-\tilde{u}(x,t)$$

on its parabolic boundary. It follows by the maximum principle that

$$1-\tilde{u}(Tx,t) \le 1-\tilde{u}(x,t) \quad \text{in } D \times (-N,0),$$

and therefore in particular

$$1 - \tilde{u}(z, t_0) \geq 1 - \tilde{u}(Tz, t_0) = 1 - \tilde{u}(x_0, t_0) \geq \frac{\lambda}{4},$$

showing (1.5.23). We can conclude, as before, that

$$1 - \tilde{u}(z, 0) \geq \alpha\lambda\, e^{-\lambda_1 N/\eta} \inf_{|x|\leq 1/4} \phi \quad \text{for } |z| \leq \tfrac{1}{4},$$

with ϕ being the solution of (1.5.21) and all the constants depending only on dimension and η. Therefore the lemma follows. □

The following result readily follows from Lemma 1.5.4 via simple rescaling.

Lemma 1.5.5. *Under the hypotheses of Theorem 1.5.2 there exist positive constants $\rho = \rho(\eta, \beta(1))$ and $\delta = \delta(\eta, \beta(1))$, depending only η, $\beta(1)$ and the dimension n, such that if u is a smooth solution of equation (1.5.2) on the unit cube Q, with $0 \leq u \leq 1$, then*

$$\max_{Q_\rho} u - \min_{Q_\rho} u =: \operatorname*{osc}_{Q_\rho} u \leq 1 - \delta.$$

We are now in position to finish the proof of Theorem 1.5.2.

Proof of Theorem 1.5.2. The proof is based on Lemma 1.5.5. For $\theta > 0$, consider the class $\mathcal{B}_\theta$ of nonlinearities β satisfying the hypotheses of Theorem 1.5.2 and $\beta(1) \leq \theta$. Let $\mathcal{A}_\theta$ denote the class of all solutions $u \in C^\infty(Q)$ of equation (1.5.2) in Q, with $\beta \in \mathcal{B}_\theta$, satisfying $0 \leq u \leq 1$. We shall show that there exists a sequence $\rho_k \downarrow 0$ for which

$$\omega_k := \sup_{u\in A_\theta} \operatorname{osc}_{Q_{\rho_k}} u \to 0 \quad \text{as } k \to \infty.$$

This readily implies (1.5.6).

To this end, set

$$\omega = \sup_{u\in A_\theta} \operatorname{osc}_Q u$$

and for $u \in \mathcal{A}_\theta$, let us define the function

$$v = \frac{u - \inf_Q u}{\omega}.$$

Then v satisfies the equation $(\tilde{\beta}(v))_t = \Delta v$, where $\tilde{\beta}(s) = \beta(s\omega + \inf_Q u)/\omega$. Moreover, by Lemma 1.5.5

$$\operatorname*{osc}_{Q_{\rho_1}} v \leq 1 - \delta$$

with $\delta = \delta(\eta, \beta(\omega + \inf_Q u)/\omega)$ and $\rho_1 = \rho(\eta, \beta(\omega + \inf_Q u)/\omega)$. Since $\inf_Q u \leq 1$ and $\omega \leq 2$, this implies that

$$\operatorname*{osc}_{Q_{\rho_1}} u \leq \left[1 - \delta(\eta, \frac{1}{\omega} C_\theta)\right] \omega$$

with

$$C_\theta = \sup\{\beta(3);\ \ \beta \in \mathcal{B}_\theta\}$$

and $\rho_1 = \rho_1(\eta, C_\theta/\omega) < 1$. We conclude that

$$\omega_1 := \sup_{u \in \mathcal{A}_\theta} \operatorname*{osc}_{Q_{\rho_1}} u \le \left[1 - \delta(\eta, \frac{1}{\omega} C_\theta)\right] \omega.$$

Now, for any $u \in \mathcal{A}_\theta$, we set

$$u_1(x,t) = u(\rho_1 x, \rho_1^2 t), \quad (x,t) \in Q.$$

The rescaled function u_1 still satisfies equation (1.5.2) in Q and therefore we can conclude, using once more Lemma 1.5.5, that

$$\omega_2 := \sup_{u \in \mathcal{A}_\theta} \operatorname*{osc}_{Q_{\rho_2}} u \le \left[1 - \delta(\eta, \frac{1}{\omega_1} C_\theta)\right] \omega_1$$

with $\rho_2 = \rho(\eta, C_\theta/\omega_1)$. Repeating the same argument, we find a decreasing sequence ρ_k such that the oscillations $\omega_k = \operatorname{osc}_{Q_{\rho_k}} u_k$, satisfy

$$\omega_k \le \left[1 - \delta(\eta, \frac{1}{\omega_{k-1}} C_\theta)\right] \omega_{k-1}.$$

This implies that the sequence $\{\omega_k\}$ is decreasing and therefore it has a limit λ. If $\lambda = 0$, then we have proven (1.5.6). If $\lambda > 0$, then the term C_θ/ω_k remains bounded in k, which implies that $\rho_k \to 0$ and that for a fixed $\tilde{\delta} \in (0,1)$, we have

$$\omega_k \le (1 - \tilde{\delta})\, \omega_{k-1}$$

showing that $\omega_k \to 0$ and contradicting that $\lambda > 0$. This finishes the proof of the theorem. □

We shall now give the proof of the general result, Theorem 1.5.1. We shall combine the ideas of De Giorgi (see [76]) and Krylov–Safonov [101].

Lemma 1.5.6. *Let $M > 0$ and $\mu_1(s) = \mu_1(-s)$ for $s < 0$. Define*

$$\hat{B}(x, M) = \frac{1}{x^2} \int_0^x \mu_1(M - s)\, s\, ds, \quad x \ge 0$$

and

$$\mu_2(x) = \frac{1}{16x^2} \int_0^x \int_0^s \mu_1(r)\, dr\, ds.$$

Then μ_2 is non-decreasing and $\hat{B}(x, M) \ge \mu_2(M)$ for $x \in [0, 4M]$.

Proof. By definition

$$\frac{d}{dx}\mu_2(x) = \frac{1}{16x^2}\int_0^x \mu_1(r)\,dr - \frac{1}{8x^3}\int_0^x\int_0^s \mu_1(r)\,drds \ge 0$$

by the convexity of $x \mapsto \int_0^x \mu_1(r)\,dr$. Also by the monotonicity of μ_1

$$\frac{d}{dx}\hat{B}(x, M) \le 0 \quad \text{for } x \in [0, M].$$

Thus by Fubini's theorem

$$\begin{aligned}\hat{B}(x, M) \ge \hat{B}(M, M) &= \frac{1}{M^2}\int_0^M \mu_1(M-s)\,sds \\ &= \frac{1}{M^2}\int_0^M\int_0^r \mu_1(l)\,dl\,ds \ge \mu_2(M)\end{aligned}$$

for $x \in [0, M]$. On the other hand, for $x \in [M, 4M]$ we have

$$\begin{aligned}\hat{B}(x, M) &= \frac{1}{x^2}\int_0^x \mu_1(M-s)\,s\,ds \\ &\ge \frac{16M^2}{x^2}\frac{1}{16M^2}\int_0^M \mu_1(M-s)\,s\,ds \ge \mu_2(M).\end{aligned}$$

□

Now, using the truncation method of De Giorgi we shall prove the next lemma.

Lemma 1.5.7. *There exists a non-decreasing $\hat{H}$ with $0 < \hat{H}(M) < M/2$, for $M > 0$, such that if $Q_{(x_0,t_0)}(R) \subseteq Q_T$, $|u| \le M$ in $Q_{(x_0,t_0)}(R)$, $0 < M < C_1$ (with C_1 as in Theorem 1.5.1) and*

$$\frac{1}{|Q_{(x_0,t_0)}(R)|}\iint_{Q_{(x_0,t_0)}(R)} (M-u)\,dxdt \le \hat{H}(M),$$

then $u \ge M/2$ in $Q_{(x_0,t_0)}(R/2)$.

Proof. Let η be a smooth function such that $0 \le \eta \le 1$ with $\eta = 0$ near $\partial_p Q(R) = \partial_p Q_{(x_0,t_0)}(R)$. Pick $k \in [0, M/2]$, and define $w = M - u \ge 0$ in $Q(R)$ so that

$$\beta'(M-w)\,\frac{\partial w}{\partial t} = \Delta w.$$

Multiplying this equation by $(w-k)^+\eta^2$ and integrating by parts over the cylinder

$B_{(x_0,R)} \times (t_0 - R^2, \tau)$, where $\tau \in (t_0 - R^2, t_0)$, we obtain

$$\begin{aligned}
&\iint \beta'(M-w)\,\frac{\partial w}{\partial t}\,(w-k)^+\eta^2\,dxdt \\
&\qquad = \iint \Delta w\,(w-k)^+\eta^2 = -\iint \nabla w\cdot\nabla\left((w-k)^+\eta^2\right) \\
&\qquad = -\iint \eta^2\left|\nabla(w-k)^+\right|^2 - 2\iint \eta\,\nabla\eta\cdot\nabla(w-k)^+\,(w-k)^+ \\
&\qquad \le -\frac{3}{4}\iint \eta^2\left|\nabla(w-k)^+\right|^2 + C\iint |\nabla\eta|^2\,\left((w-k)^+\right)^2.
\end{aligned}$$

We can replace $\eta^2\left|\nabla(w-k)^+\right|^2$ by $\left|\nabla\left(\eta\,(w-k)^+\right)\right|^2$ because the error is equal to $|\nabla\eta|^2\left((w-k)^+\right)^2$. Thus

$$\begin{aligned}
&\iint \beta'(M-w)\,\frac{\partial w}{\partial t}\,(w-k)^+\eta^2 \\
&\qquad \le -\frac{3}{4}\iint \left|\nabla\left(\eta(w-k)^+\right)\right|^2 + C\iint \left|\nabla\eta^2\right|((w-k)^+)^2.
\end{aligned} \tag{1.5.24}$$

Let us denote

$$B(x) = \int_0^x \beta'(M-k-s)\,s\,ds$$

so that

$$\frac{\partial}{\partial t}B\left((w-k)^+\right) = \beta'(M-w)\,(w-k)^+\frac{\partial w}{\partial t}$$

and also

$$B\left((w-k)^+\right) \ge \left((w-k)^+\right)^2\ \hat{B}\left((w-k)^+, M-k\right)$$

by the definition of $\hat{B}$. Since $k \le M/2$, $u < M$ and $w = M-u$, it follows that $(w-k)^+ \in [0, 4(M-k)]$, so by Lemma 1.5.6 and the above inequality

$$B\left((w-k)^+\right) \ge \left((w-k)^+\right)^2\ \mu_2(M-k) \ge \left((w-k)^+\right)^2\ \mu_2(M/2).$$

We also have

$$\begin{aligned}
B((w-k)^+) &\le (w-k)^+\int_0^{(w-k)^+}\beta'(M-k-s)\,ds \\
&\le (w-k)^+\,[\beta(M-k)-\beta(M-w)] \le C\,\left[(w-k)^+\right]^2.
\end{aligned}$$

Let

$$I = \iint \beta'(M-w)(w-k)^+\,\frac{\partial w}{\partial t}\,\eta^2\,dxdt.$$

Then

$$
\begin{aligned}
I &= \iint \frac{\partial}{\partial t} B\left((w-k)^+\right) \eta^2 \, dxdt \\
&= \int_{B_R(x_0)} B\left((w-k)^+\right) \eta^2|_{t=\tau} dx - 2 \iint B\left((w-k)^+\right) \eta^2 \tfrac{\partial \eta}{\partial t} \\
&\geq \mu_2(M/2) \int \left((w-k)^+\right)^2 \eta^2|_{t=\tau} - C \iint ((w-k)^+)^2 \, \eta \left|\tfrac{\partial \eta}{\partial t}\right|.
\end{aligned}
\tag{1.5.25}
$$

Without loss of generality we can assume $\mu_2(M/2) \leq 3/4$. Hence, combining (1.5.24), (1.5.25) we obtain

$$
\begin{aligned}
&\mu_2(M/2)\left[\int_{B_R(x_0)} \left((w-k)^+ \eta\right)^2 |_{t=\tau} + \iint \left|\nabla\left(\eta\,(w-k)^+\right)\right|^2\right] \\
&\qquad \leq C \left[\iint |\nabla \eta|^2 \left((w-k)^+\right)^2 + \iint ((w-k)^+)^2 \, \eta \left|\tfrac{\partial \eta}{\partial t}\right|\right] \leq C'.
\end{aligned}
$$

From the Sobolev Embedding Theorem, inequality (1.5.4), mentioned at the beginning of the section, we obtain

$$
\begin{aligned}
\left\|(w-k)^+ \eta\right\|_{L^{\frac{2(n+2)}{n}}} &\leq C \left\|(w-k)^+ \eta\right\|_{\overset{\circ}{V}_2(Q(R))} \\
&\leq \frac{C}{\mu_2(M/2)} \left[\iint_{\{w \geq k\} \cap \operatorname{supp} \eta} |\nabla \eta|^2 + \eta \, |\eta_t|\right]^{1/2}.
\end{aligned}
\tag{1.5.26}
$$

We start now the iteration process. For $m \in \mathbb{N}$ let

$$
\begin{aligned}
k_m &= \frac{M}{2}\left(1 - \frac{1}{2^m}\right), \\
R_m &= \frac{R}{2}\left(1 + \frac{1}{2^m}\right), \\
Q_m &= Q(R_m),
\end{aligned}
$$

and

$$
J_m = \frac{1}{|Q_0|} \iint_{Q_m} \left[(w-k_m)^+\right]^2 \, dxdt.
$$

In particular,

$$
J_0 = \frac{1}{|Q_0|} \iint_{Q(R_0)} (M-u) \, dxdt.
$$

In order to prove the lemma it suffices to show that

$$
\lim_{m \to \infty} J_m = 0 \tag{1.5.27}
$$

provided $J_0 < \hat{H}(M)$.

Let $\eta_m \in C_0^\infty$ such that $\eta_m = 1$ on Q_{m+1} with $\eta_m = 0$ near $\partial_p Q_m$, and $|\nabla \eta_m|^2$, $|\partial \eta_m/\partial t| \le 4^m/R^2$. Thus we have from Hölder inequality and (1.5.26) that

$$\begin{aligned}
|Q_0| \cdot J_{m+1} &= \iint_{Q_{m+1}} \left[(w-k_{m+1})^+\right]^2 dxdt \\
&\le \left(\iint_{Q_m} \left((w-k_{m+1})^+ \eta_m\right)^{2(n+2)/n} dxdt\right)^{\frac{n}{n+2}} |Q_m \cap \{w \ge k_{m+1}\}|^{\frac{2}{n+2}} \quad (1.5.28)\\
&\le \frac{C}{\mu_2(M/2)^2} \frac{4^m}{R^2} |Q_m \cap \{w \ge k_{m+1}\}|^{1+\frac{2}{n+2}} .
\end{aligned}$$

But

$$\iint_{Q_m} \left[(w-k_m)^+\right]^2 \ge (k_{m+1}-k_m)^2 \cdot |Q_m \cap \{w \ge k_{m+1}\}|$$

so that

$$|Q_m \cap \{w \ge k_{m+1}| \le C\, R^{n+2}\, 4^m M^2 J_m. \tag{1.5.29}$$

By combining (1.5.28), (1.5.29) it follows that for $R \le 1$

$$\begin{aligned}
J_{m+1} &\le \frac{C}{\mu_2(M/2)^2} \frac{4^m}{R^{n+2}} \left(\frac{R^{n+2} 4^m J_m}{M^2}\right)^{1+2/(n+2)} \\
&\le \frac{C}{\mu_2(M/2)^2} \left(4^{2+2/(n+2)}\right)^m \frac{1}{M^{2+4/(n+2)}} J_m^{1+2/(n+2)}.
\end{aligned}$$

Thus it is easy to see that $J_m \to 0$ provided

$$J_0 \le C^{**} = C^* M^{n+4} \left(\mu_2(M/2)\right)^{(n+2)} ,$$

which completes the proof of the lemma. □

Corollary 1.5.8. *There exists a non-decreasing function $H(M)$ with $0 \le H(M) \le M/2$, such that if $Q(R) \subseteq Q_T$, $|u| \le M$ in $M(R)$ for $0 < M < C_1$, and $u(x,t) \le M/2$ for $(x,t) \in Q(R/2)$, then*

$$|Q(R) \cap \{u \le M - H(M)\}| \ge H(M)\ |Q(R)| .$$

Proof. Let $H(M) = \hat{H}(M)/(2C_1 + 1)$, with $\hat{H}(M)$ as in the previous lemma. We shall argue by contradiction. Assume that the result were false. Write

$$\iint_{Q(R)} (M-u)\, dx\, dt = \iint_{A_1} + \iint_{A_2} \tag{1.5.30}$$

where

$$\begin{aligned}
A_1 &= Q(R) \cap \{u \le M - H(M)\} \\
A_2 &= Q(R) \cap \{u > M - H(M)\}.
\end{aligned}$$

Therefore

$$\iint_{Q(R)} (M-u)\,dx\,dt \le 2\,C_1\,|Q(R)\cap\{u\le M-H(M)\}| + H(M)\,|Q(R)|$$
$$\le \hat{H}(M)|Q(R)|.$$

Thus using the past lemma $u \ge M/2$ in $Q(R/2)$, which contradicts the hypothesis. □

Fix $\varepsilon > 0$, $m > 0$ and denote by $g = g_{\varepsilon,m} \in C^\infty$ a function such that $g \ge 0$, $g'' \ge 0$, $0 \le g' \le 1$ and $[s-\varepsilon-1/m]^+ \le g(s) \le [s-\varepsilon]^+$.

Lemma 1.5.9. *Let $z(x,t) = g\,(u(x,t))$. Then*

$$z_t - a(x,t)\Delta z \le 0 \tag{1.5.31}$$

where $\delta \le a(x,t) \le \delta^{-1}$, $\delta = \delta(\varepsilon, \mu_1(\varepsilon), \hat{\mu}_1(\varepsilon), C_1, n)$.

Proof. We have

$$\frac{\partial z}{\partial t} = g'(u)\frac{\partial u}{\partial t}$$

and

$$\beta'(u)\,\frac{\partial u}{\partial t} - \Delta u = 0.$$

Also,

$$\Delta z = g''(u)\,|\nabla u|^2 + g'(u)\,\Delta u \ge g'(u)\,\Delta u.$$

Define $\bar{u}(x,t) \equiv \max\{u(x,t), \varepsilon\}$ and

$$a(x,t) = \frac{1}{\beta'(\bar{u}(x,t))}.$$

From our assumptions on β and u, there exists $\delta > 0$ such that

$$\delta \le a(x,t) \le \delta^{-1}.$$

If $u \le \varepsilon$, then $z = g(u) = 0$ and $\partial z/\partial t = \Delta z = \nabla z = 0$ almost everywhere. Thus (1.5.31) holds. If, on the other hand, $u > \varepsilon$ then $a(x,t) = 1/\beta'(u)$, and $\beta'(u)\,\partial z/\partial t - \Delta z \le 0$ with $\beta'(u) > 0$. Therefore (1.5.31) also holds in this case. □

The following result is due to Krylov and Safonov [101]. We refer the reader to their paper for its proof.

Theorem 1.5.10. *Let w be a non-negative function in Q_R such that*

$$\frac{\partial w}{\partial t} - a_{ij}(x,t)\,\frac{\partial^2 w}{\partial x_i \partial x_j} \ge 0 \quad \textit{in } Q_R.$$

Assume that the coefficients a_{ij} satisfy

$$\delta\ |\xi|^2 \leq a_{ij}(x,t)\,\xi_i\xi_j \quad \textit{for all } (x,t) \in Q_R \tag{1.5.32a}$$

and

$$\|a_{ij}\|_{L^\infty Q_R} \leq \delta^{-1}, \tag{1.5.32b}$$

and that

$$|Q(R) \cap \{w > 1\}| \geq \theta_0\ |Q(R)|$$

for some $\theta_0 \in (0,1)$. Then there exists $a_0 = a_0(\delta, n, \theta_0)$ such that

$$w(x, t_0) \geq a_0 \quad \textit{for } |x - x_0| \leq R/2.$$

Corollary 1.5.11. *Under the hypotheses of the above theorem, if*

$$k_n^2 = \min\left\{1/5;\ \inf_{\theta\in[0,1]}\left(\frac{1-(1-\theta/2)^{2/(n+2)}}{\theta^2}\right)\right\}$$

there exists $\tilde{a}_0 = \tilde{a}_0(\delta, n, \theta_0)$ such that

$$w(x,t) \geq \tilde{a}_0 \quad \textit{for all } (x,t) \in Q(k_n\theta_0 R).$$

Proof. Let $\alpha = k_n\theta_0$ and pick any $t \in \left[t_0 - (\alpha R)^2, t_0\right]$. Then

$$Q_{(x_0,t)}(\sqrt{1-\alpha^2}R) \subseteq Q_{(x_0,t_0)}(R).$$

We also have that

$$\begin{aligned}
&|Q_{(x_0,t_0)}(R) \setminus Q_{(x_0,t)}(\sqrt{1-\alpha^2}\,R)| \\
&\qquad \leq (1-(1-\alpha^2)^{(n+2)/2})\left|Q_{(x_0,t_0)}(R)\right| \leq \frac{\theta_0}{2}\left|Q_{(x_0,t_0)}(R)\right|
\end{aligned}$$

since

$$1-\alpha^2 = 1 - k_n^2\theta_0^2 \geq 1 - \left(1 - \left(1 - \frac{\theta_0}{2}\right)^{2/(n+2)}\right) = \left(1 - \frac{\theta_0}{2}\right)^{2/(n+2)}$$

and

$$1-(1-\alpha^2)^{(n+2)/2} = \left(1 - \left(1 - \frac{\theta_0}{2}\right)\right) = \frac{\theta_0}{2}.$$

Thus

$$\left|Q_{(x_0,t)}(\sqrt{1-\alpha^2}\,R) \cap \{w \geq 1\}\right| \geq \frac{\theta_0}{2}\left|Q_{(x_0,t)}(R)\right| \geq \frac{\theta_0}{2}\left|Q_{(x_0,t)}(\sqrt{1-\alpha^2}\,R)\right|$$

and this for all $t \in [t_0 - (\alpha R)^2, t_0]$. Thus by the Theorem 1.5.10 there exists $\tilde{a}_0$ depending only on n, δ, and θ_0 such that

$$w(x,t) \geq \tilde{a}_0 \quad \text{for } |x - x_0| < \frac{\sqrt{1-\alpha^2}\,R}{2}$$

for all $t \in [t_0 - (\alpha R)^2, t_0]$. Since $\alpha \le 1/\sqrt{5}$, then $\alpha \le \sqrt{1-\alpha^2}/2$. Therefore

$$Q_{(x_0,t_0)}(\alpha R) \subseteq B_{x_0}(\sqrt{1-\alpha^2}\, R/2) \times [t_0 - (\alpha R)^2, t_0]$$

which proves the corollary. □

Corollary 1.5.12. *Suppose that $z(x,t)$ verifies*

(i) $z \le M$ *in* $Q(R)$;

(ii) $\frac{\partial z}{\partial t} - a_{ij}\frac{\partial^2 z}{\partial x_i \partial x_j} \le 0$ *in* $Q(R)$;

(iii) $|Q(R) \cap \{z \le M/2\}| \ge \theta_0 |Q(R)|$ *for some* $\theta_0 \in (0,1)$.

Then there exists a_0 depending only on n, δ, and θ_0 such that

$$z(x,t) \le M - \frac{Ma_0}{2} \quad \textit{in } Q(k_n\theta_0 R).$$

Proof. Set

$$w(x,t) = \frac{2}{M}(M - z(x,t)).$$

Then

(i) $w \ge 0$,

(ii) $\frac{\partial w}{\partial t} - a_{ij}\frac{\partial^2 w,}{\partial x_i \partial x_j} \ge 0$ in $Q(R)$,

(iii) $|Q(R) \cap \{w \ge 1\}| \ge \theta_0 |Q(R)|$.

Thus by the previous corollary there exists $a_0 = a_0(n, \delta, \theta_0)$ such that

$$w(x,t) \ge a_0 \quad \text{in } Q_{(x_0,t_0)}(k_n\theta_0 R)$$

which implies that

$$z(x,t) \le M - \frac{Ma_0}{2} \quad \text{in } Q_{(x_0,t_0)}(k_n\theta_0 R).$$ □

Let $Q_T = \Omega \times (0,T)$, $Q \subset\subset Q_T$, and $d = \operatorname{dist}(Q, \partial_p Q_T)$. For $R < d$ consider $Q(R) = Q_{(x_0,t_0)}(R)$ with $(x,t) \in Q$. For the proof of Theorem 1.5.1 we shall also need the following lemma.

Lemma 1.5.13. *Given $M, \varepsilon, \theta_0, m$ and $z = g_{\varepsilon,m}(u)$ (as in Lemma 1.5.9), there exist R^* and σ^* depending only on M, ε, θ_0, and the "data" (i.e., μ_1, $\hat{\mu}_1$, c_1) such that*

$$0 < R^* \le d/c, \quad 0 < \sigma^* < M/2$$

and for $R < R^*$

(i) *$z \le M$ in $Q(R)$;*

(ii) *if $|Q(R) \cap \{z = 0\}| \ge \theta_0 \, |Q(R)|$, then $z \le M - \sigma^*$ in $Q(k_n \theta_0 R)$.*

Proof. We observe that z verifies the conditions in Corollary 1.5.12 with δ depending only on ε and the data (see Lemma 1.5.9). Let d_0 be the constant in Corollary 1.5.12 which depends only on n, δ and θ_0. Setting $R^* = R/2$ and $\theta^* = \min\{M/2, Ma_0/2\}$ we have

$$z(x,t) \le M - \frac{Ma_0}{2} \le M - \sigma^* \quad \text{in } Q(k_n \theta_0 R).$$ □

Proposition 1.5.14. *Let $(x_0, t) \in Q$ such that $u(x_0, t_0) = 0$. Then there exist $M_k \downarrow 0$ and $R_k \downarrow 0$ as $k \uparrow \infty$ depending only on the "data" such that*

$$|u(x,t)| \le M_k \quad \textit{in } Q_{(x_0,t_0)}(R_k).$$

Proof. With the same notation as in Lemma 1.5.13 let

$$\sigma(M) = \sigma^* \left(H(M), M - H(M), H(M) \right)$$

and

$$R^*(M) = R^* \left(H(M), M - H(M), H(M) \right).$$

We have

$$0 < \sigma(M) \le \frac{H(M)}{2}$$

and

$$0 < R^*(M) \le \frac{d}{2} \quad \text{for } M \in (0, C_1].$$

Let

$$M_1 = c_1, \quad M_{k+1} = M_k - \sigma(M_k)$$
$$R_1 = R(M), \quad R_{k+1} = \min\{R(M_{k+1}), k_n \, H(M_k) \, R_k\}.$$

Claim. $|u(x,t)| \le M_k$ in $Q_{(x_0,t_0)}(R_k)$.

We shall use induction:

(1) For $k = 1$, $|u| \le c_1 = M_1$ in $Q_T \supset Q_{(x_0,t_0)}(R_1)$.

(2) Assuming that

$$|u| \le M_k \quad \text{on } Q(R_k)$$

we shall prove that

$$|u| \le M_{k+1} \quad \text{on } Q(R_{k+1}).$$

Set $\varepsilon = \varepsilon(M_k) = M_k - H(M_k)$. For $m > 0$ define $z(x,t) = g_{\varepsilon,m}(u)$. Thus $z \le [u - \varepsilon(M_k)]^+ \le H(M_k)$ in $Q(R_k)$. Also since $u(x_0,t_0) = 0$

$$\begin{aligned} |Q(R_k) \cap \{z = 0\}| &\ge |Q(R_k) \cap \{u \le \varepsilon(M_k)\}| \\ &= |Q(R_k) \cap \{u \le M_k - H(M_k)\}| \\ &\ge H(M_k)\,|Q(R_k)| \end{aligned}$$

by Corollary 1.5.8. Hence, by the previous lemma

$$z \le H(M_k) - \sigma(M_k) \quad \text{in } Q(k_n H(M_k) R_k) \supset Q(R_{k+1}).$$

By the triangle inequality

$$\begin{aligned} |u| &\le \left[u - \varepsilon(M_k) - \frac{1}{m}\right]^+ + \varepsilon(M_k) + \frac{1}{m} \le z + \varepsilon(M_k) + \frac{1}{m} \\ &\le H(M_k) - \sigma(M_k) + \varepsilon(M_k) + \frac{1}{m} = M_{k+1} + \frac{1}{m} \le M_{k+1} \end{aligned}$$

(since m is arbitrary in the g construction). Thus

$$|u| \le M_{k+1} \quad \text{in } Q(R_{k+1}).$$

Now it is easy to see that $M_k \downarrow 0$ and $R_k \downarrow 0$ as $k \to \infty$ which finishes the proof of the proposition. □

Proposition 1.5.15. *Let M_k and R_k be as in the previous proposition. Let $(x_0,t_0) \in Q$ such that* $\operatorname{dist}((x_0,t_0), \partial_p Q) > 2d$, *where* $d = \operatorname{dist}(Q, \partial_p Q_T)$. *Suppose that*

$$M_{k_0+1} < u(x_0,t_0) \le M_{k_0} \quad \textit{for some } k_0.$$

Then

$$u \ge \frac{M_{k_0}}{2} \quad \textit{in } Q^1_{(x_0,t_0)}(\bar{R}_{k_0}) \equiv Q' \tag{1.5.33}$$

where

$$\bar{R}_{k_0} = \min\left\{R_{k_0+1}; \tfrac{R_{k_0}}{2}\right\}$$

and

$$Q^1_{(x_0,t_0)}(R) = \{(x,t) : |x - x_0| < R,\ |t - t_0| < R^2\}.$$

Proof. Suppose that (1.5.33) does not hold. Then there exists $(x_1,t_1) \in Q'$ such that

$$u(x_1,t_1) < \frac{M_{k_0}}{2}.$$

If $t_1 \le t_0$, then by using induction as in the proof of the previous proposition we obtain

$$|u| \le M_{k_0+1} \quad \text{in } Q_{(x_0,t_0)}(R_{k_0+1})$$

which implies that $u(x_0, t_0) \le M_{k_0+1}$, impossible. Thus we have that $u \ge \frac{M_{k_0}}{2}$ in $Q' \cap \{t \le t_0\}$.

Suppose now that $u(x_1, t_1) < M_{k_0}/2$, for some $t_1 > t_0$. It then follows that $(x_0, t_0) \in Q_{(x_0,t_1)}(R_{k_0+1})$. Using once more the induction argument as in Proposition 1.5.14 together with $u(x_1, t_1) \le M_{k_0}/2$ and Lemma 1.5.7 we obtain that $|u| \le M_{k_0+1}$ in $Q_{(x_0,t_1)}(R_{k_0+1})$, i.e.,

$$u(x_0, t_0) \le M_{k_0+1} \quad (\text{since } (x_0, t_0) \in Q_{(x_0,t_1)}(R_{k_0+1}))$$

which is a contradiction. Thus we have that

$$u \ge \frac{M_{k_0}}{2} \quad \text{in } Q^1_{(x_0,t_1)}(\bar{R}_{k_0})$$

which proves the proposition. □

Proof of Theorem 1.5.1. Fix $\varepsilon > 0$ and $(x_0, t_0) \in Q$. We must find η depending only on ε and the "data" such that

$$\text{if } |(x_0, t_0) - (x_1, t_1)| < \eta, \text{ then } |u(x_0, t_0) - u(x_1, t_1)| < \varepsilon.$$

Call $L = u(x_0, t_0)$, and without loss of generality we can assume that $L \ge 0$, otherwise we replace u by $-u$. We shall consider two cases:

Case 1. $L \le \varepsilon/3$.

Claim. There exist $\eta_1 > 0$ *depending only on* ε *and the "data" such that*

$$|u(x_1, t_1)| < \frac{2\varepsilon}{3} \quad \textit{for } (x_1, t_1) \in B_{(x_0,t_0)}(\eta_1).$$

To prove this we use the induction method of Proposition 1.5.15. Without loss of generality we can assume that $\varepsilon/3 < M_1/2$. Then by using that $M_k \downarrow 0$ we get that if

$$\frac{M_{k+1}}{2} \le \frac{\varepsilon}{3} < \frac{M_k}{2}$$

then $|u| \le M_{k+1} < 2\varepsilon/3$ in $Q_{(x_0,t_0)}(R_{k+1})$. By the same argument given in the proof of Proposition 1.5.15 it follows that there exists $\eta > 0$ such that

$$|u| \le \frac{2\varepsilon}{3} \quad \text{in } B_{(x_0,t_0)}(\eta).$$

which proves the result.

Case 2. $L > \varepsilon/3$.

Claim. There exists $\eta_2 > 0$ *such that*

$$u(x,t) > \frac{\varepsilon}{6} \quad \textit{in } (x,t) \in B_{(x_0,t_0)}(\eta_2).$$

Indeed, by case 1, there exists $\eta_2 > 0$ such that if for some (x_1, t_1), $u(x_1, t_1) \leq \varepsilon/6$, then $u(x,t) \leq \varepsilon/3$ for every $(x,t) \in B_{(x_1,t_1)}(\eta_2)$ (notice that η_2 is independent of (x_1, t_1)). Now let $(x,t) \in B_{(x_0,t_0)}(\eta_2)$. If $u(x,t) < \varepsilon/6$ then $u(x_0, t_0) \leq 2\varepsilon/6 = \varepsilon/3$ which contradicts our hypothesis. Thus $u(x,t) \geq \varepsilon/6$ for any $(x,t) \in B_{(x_0,t_0)}(\eta)$. But in this neighborhood of (x_0, t_0) u satisfies the equation

$$\frac{\partial u}{\partial t} = a(x,t)\Delta u.$$

where$a(x,t) = \frac{1}{\beta'(u(x,t))}$ with $c_1 \leq a(x,t) \leq c_2$.

By the Krylov–Safonov Theorem, $u|_{B_{(x_0,t_0)}(\eta)}$ is Hölder continuous with exponent and constants depending only on ε and the “data”. In particular we obtain the desired result. □

1.6 Existence of weak solutions

This section is concerned with the existence of weak solutions of the equation

$$\frac{\partial u}{\partial t} = \Delta\varphi(u) \tag{1.6.1}$$

with $\varphi \in \Gamma_a$. We remind the reader that the class of nonlinearities Γ_a is defined by the following conditions:

(i) $\varphi\colon [0,\infty) \to \mathbb{R}$ is a continuous non-negative function with $\varphi(0) = 0$.

(ii) there exist $a \in (0,1)$ such that for all $u > 0$

$$a \leq \frac{u\,\varphi'(u)}{\varphi(u)} \leq a^{-1}.$$

We start by considering the existence problem in finite cylinders. Let $Q_T = \Omega \times (0,T]$, where $\Omega \subset \mathbb{R}^n$ is an open bounded set. As in the previous section we denote by β the inverse function of φ. Thus $\beta\colon [0,\infty) \to [0,\infty)$.

We shall study the following initial Dirichlet problem (IDP):

$$\left.\begin{aligned} (\beta(v))_t &= \Delta v && \text{in } Q_T,\\ \beta\,(v(x,0)) &= u_0(x) && x \in \Omega,\\ v(x,t) &= 0 && (x,t) \in \partial\Omega \times (0,T]. \end{aligned}\right\} \tag{1.6.2}$$

We assume that the nonlinearity β satisfies the following conditions:

(a) $\beta : \mathbb{R} \to \mathbb{R}$ is locally absolutely continuous.

(b) β strictly increasing with $\beta(0) = 0$.

(c) there exist μ, $\hat{\mu}$ and $s_0 > 0$, such that

  (i) μ is increasing on $[0, s_0]$ and decreasing on $[s_0, \infty)$ with $\int_{s_0}^{\infty} \mu(s)ds = \infty$,

  (ii) $\hat{\mu}$ is decreasing on $[0, s_0]$ with $\int_0^{s_0} \hat{\mu}(s)\, ds < \infty$,

  (iii) $0 < \mu(|s|) \le \beta'(s) \le \hat{\mu}(|s|)$, for any s.

Remark. It is easy to verify that if $\varphi \in \Gamma_a$, then $\beta = \varphi^{-1}$ satisfies the hypotheses (a), (b) and (c). In particular, in the case where $\varphi(u) = u^m$, $m > 0$, it follows that $\beta(u) = u^{1/m}$. Hence (a)–(c) are immediate.

Notation. We introduce the following notation:

$$
\begin{aligned}
&W^{1,1}(Q_T) = \Big\{ u \in L^2(Q_T) : \nabla u, \frac{\partial u}{\partial t} \in L^2(Q_T) \Big\},\\
&\overset{\circ}{W}{}^{1,1}(Q_T) = \text{closure}\,\{ \eta \in C^\infty(Q_T) : \eta \equiv 0 \text{ on } \partial Q_T \} \subset W^{1,1}(Q_T),\\
&E = \{ \psi \in \overset{\circ}{W}{}^{1,1}(Q_T) : \psi(x, T) = 0 \},\\
&\overset{\circ}{V}_2(Q_T) = L^\infty\big((0, T]; L^2(\Omega)\big) \cap L^2\big((0, T]; H_0^1(\Omega)\big).
\end{aligned}
$$

Definition. A function $v \in \overset{\circ}{V}_2\, (Q_T)$ is called a solution of the IDP (1.6.2) if $\beta(v) \in L^2(Q_T)$ and

$$
\iint_{Q_T} \Big(\beta(v) \frac{\partial \psi}{\partial t} - \nabla v \cdot \nabla \psi \Big)\, dxdt = \int_\Omega u_0(x)\, \psi(x, 0)\, dx
$$

for all $\psi \in E$.

Theorem 1.6.1. *For any* $u_0 \in L^\infty(\Omega)$ *there exists a solution* $v \in L^\infty(Q_T) \cap C(Q_T)$ *of the IDP* (1.6.2).

Proof. Let J_j^n denote an n-dimensional mollifier which is positive, symmetric, with $\int J_j^n = 1$ and $\operatorname{supp} J_j^n \subseteq B_{1/j}(0)$. Define

$$
\bar{u}_0(x) = \begin{cases} u_0(x) & x \in \Omega, \\ 0 & \text{otherwise,} \end{cases}
$$

and

$$
u_{0j} = \big(J_j^n * \bar{u}_0 \big) \cdot \phi_j
$$

where ϕ_j is an appropriate smooth cut-off function such that

$$
\begin{aligned}
&u_{0j} \in C_0^\infty(\Omega),\\
&u_{0j} \to u_0 \quad \text{in } L^p(\Omega), \text{ for any } p < \infty
\end{aligned}
$$

and

$$\|u_{0j}\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} .$$

Also define

$$\varphi_j = \beta_j^{-1}$$

where

$$\beta_j = \big(J_j^1 * \beta\big)(s) - (J_j^1 * \beta)(0).$$

It is clear that

(i) $\beta_j \in C_0^\infty(\mathbb{R})$,

(ii) $\beta_j(0) = 0$,

(iii) $\beta_j'(s) = J_j^1 * \beta'(s) > 0$,

since $\beta(\cdot)$ is locally absolutely continuous. Hence we have that

$$\varphi_j \in C^\infty(\mathbb{R}),$$

and for any $M > 0$ there exists $\varepsilon = \varepsilon(j, M) > 0$ such that

$$\varphi_j'(s) \ge \varepsilon(j, M) \quad \text{for } |s| \le M.$$

By the classical theory of non-degenerate quasilinear equations (see [103]) we conclude that the IDP

$$\left.\begin{aligned} (u_j)_t &= \Delta\varphi_j(u_j) && \text{in } Q_T,\\ u_j(x,0) &= u_{0j} && x \in \Omega,\\ u_j(x,t) &= 0 && \text{on } \partial\Omega \times (0,T], \end{aligned}\right\} \tag{1.6.3}$$

has a classical solution u_j. By the maximum principle $u_j \in L^\infty(Q_T)$ with

$$\|u_j\|_{L^\infty(Q_T)} \le \|u_{0j}\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} .$$

Therefore $\{u_j\}_0^\infty$ is uniformly bounded. Set $v_j = \varphi_j(u_j)$.

We claim that $\{v_j\}$ is uniformly bounded. Indeed, since the φ_j's are increasing and the $\{u_j\}$ is uniformly bounded it suffices to show that $\{\varphi_j(r)\}_{j=0}^\infty$ is bounded in j for all $r > 0$. To this end, let $r > 0$ and set $\sigma_j = \varphi_j(r)$ so that $r = \beta_j(\sigma_j)$. Thus

$$r = \int_0^{\sigma_j} \beta_j'(s)ds = \int_0^{\sigma_j} J_j^1 * \beta'(s)ds.$$

If $\sigma_j \ge s_0 + 1$ then

$$\begin{aligned} J_j^1 * \beta'(s) &= \int_{-1/j}^{1/j} J_j^1(t)\,\beta'(s-t)\,dt \\ &\ge \int_{-1/j}^{1/j} J_j^1(t)\mu(s-t)\,dt \ge \mu(s-1) \end{aligned}$$

and therefore $r \geq \int_{s_0+1}^{\sigma_j} \mu(s-1)\,ds$. Thus if $\{\varphi_j(r)\}$ is not bounded it follows that $\sigma_j \uparrow \infty$ as $j \uparrow \infty$, i.e.,

$$r \geq \int_{s_0+1}^{\infty} \mu(s-1)\,ds = \int_{s_0}^{\infty} \mu(s)\,ds = \infty.$$

a contradiction. Thus we have established that $\{v_j\}$ is uniformly bounded.

Now we shall show that $\{v_j\}$ is an equicontinuous family. From the results in Section 1.5, it suffices to prove that $\{\beta_j\}$ satisfies uniformly on j the assumptions of Theorem 1.5.1. Thus we have to show that there exist μ_1 and $\hat{\mu}_1$ (independent of j) such that

$$0 < \mu_1(\delta) \leq \beta_j'(s) \leq \hat{\mu}_1(\delta) \quad \text{for } 0 < \delta < |s| \leq C_1 \tag{1.6.4}$$

where C_1 is an upper bound for $\|u_j\|_{L^\infty(Q_T)}$ and $\|\beta(u_j)\|_{L^\infty(Q_T)}$.

Let μ and $\hat{\mu}$ as in condition (c) defining the class of nonlinearities β, with $s_0 = C_1 + 1$. We shall show (1.6.4) for $s > 0$. The case $s < 0$ can be treated the same way. Define

$$\begin{aligned}
\mu_1(s) &= \frac{1}{C_1+1} \int_0^s \min\{\mu(r), \mu(C_1+1)\}\,dr \\
&\leq \frac{s}{C_1+1} \min\{\mu(s), \mu(C_1+1)\} \\
&\leq \min\{\mu(s), \mu(C_1+1)\} \leq \beta'(s)
\end{aligned}$$

for $s \leq C_1$. Since $\mu(\cdot)$ is increasing, it follows that $\mu_1(\cdot)$ is convex and thus for $0 < \delta \leq |s| \leq C_1$ we obtain

$$\beta_j' = J_j^1 * \beta'(s) \geq J_j^1 * \mu_1(s) \geq \mu_1(s) \geq \mu_1(\delta).$$

Defining

$$\hat{\mu}^*(s) = \max\{\,\hat{\mu}(s), \hat{\mu}(C_1+1)\,\}$$

$$j_0 = \left[\frac{2}{s}\right] + 1$$

(where $[\cdot]$ denotes the integer part), for $j \geq j_0$ it follows that

$$\beta_j'(s) = J_j^1 * \beta'(s) \leq J_j^1 * \hat{\mu}(s) \leq \hat{\mu}\left(\frac{s}{2}\right)$$

since

$$J_j^1 * \hat{\mu}(s) = \int_{-1/j}^{1/j} J_j^1(t)\hat{\mu}(s-t)\,dt$$

and

$$|s-t| \geq s - 1/j \geq s - 1/j_0 \geq s/2$$

with $\hat{\mu}$ decreasing in $[0, s_0]$. Thus

$$\hat{\mu}(s-t) \leq \hat{\mu}\left(\frac{s}{2}\right) \leq \hat{\mu}^*\left(\frac{s}{2}\right).$$

For $j < j_0$ we have

$$\beta_j'(s) \le J_j^1 * \hat{\mu}(s) \le J_{j_0}^1 * \hat{\mu}^*(0) < \infty.$$

Set

$$\hat{\mu}_1(s) = \max\left\{ \hat{\mu}^*\left(\tfrac{s}{2}\right);\, J^1_{[\frac{2}{j}]+1} * \hat{\mu}^*(0) \right\}$$

(since $\hat{\mu}$ is decreasing). Thus $\hat{\mu}_1$ is decreasing and

$$\beta_j'(s) \le \hat{\mu}_1(s) \le \hat{\mu}_1(\delta) \quad \text{for } 0 < \delta \le |s| \le C_1.$$

We have shown above that there exist μ_1 and $\hat{\mu}_1$ as in the assumption of the equicontinuity Theorem 1.5.1 (since the μ_1, $\hat{\mu}_1$ are independent of j).

Now $\{v_j\}$ is equicontinuous and uniformly bounded, therefore there exists a subsequence $\{v_{j_k}\}$ such that

$$v_{j_k} \to v \quad \text{uniformly on compacts.}$$

We shall show next that v is a solution of the IDP (1.6.2). In fact, it is clear that

(i) $v \in L^\infty(Q_T)$ with

$$\|v\|_{L^\infty(Q_T)} \le \sup_{j_k} \|v_{j_k}\|_{L^\infty(Q_T)} < C_1.$$

(ii) $v \in C(Q_T)$.

(iii) Also, $v \in \overset{\circ}{V}_2(Q_T)$.

Indeed, from i) it follows that $v \in L^\infty\big((0,T) : L^2(\Omega)\big)$. From the equation

$$\iint_{Q_T} \left(\beta_{j_k}(v_{j_k}) \frac{\partial \psi}{\partial t} - \nabla v_{j_k} \cdot \nabla \psi \right) dx dt + \int_\Omega u_{0j}\, \psi(x,0)\, dx = 0 \tag{1.6.5}$$

it follows that

$$\iint_{Q_T} \left|\nabla v_{j_k}\right|^2 dx dt \le M. \tag{1.6.6}$$

The formal proof of (1.6.6) follows by an argument similar to that given in Section 1.2.

Thus $\{v_{j_k}\}$ are bounded in $L^2\big((0,T); H_0^1(\Omega)\big)$. Therefore, there exists a subsequence (with the same notation) $\{v_{j_k}\}$ such that

$$v_{j_k} \to w \quad \text{weakly in } L^2\big((0,T); H_0^1(\Omega)\big).$$

Since $v_{j_k} \to v$ in $L^2\big((0,T); L^2(\Omega)\big)$ it follows that $v = w$ and $v \in L^2\big((0,T); H_0^1(\Omega)\big)$. Taking $j \to \infty$ in the equation (1.6.5) and using that

(i) $\beta_{j_k} \to \beta$ uniformly on compacts,

(ii) $v_{j_k} \to v$ uniformly on compacts and is uniformly bounded.

(iii) $\nabla v_{j_k} \to \nabla u$ weakly in L^2

we obtain that

$$\iint_{Q_T} \left(\beta(v)\, \frac{\partial \psi}{\partial t} - \nabla v \cdot \nabla \psi \right) dxdt + \int_{\Omega} u_0(x)\, \psi(x,0)\, dx = 0$$

which completes the proof of Theorem 1.6.1. □

To finish the section let us mention how one may extend this result on $\mathbb{R}^n$, with $v_0 \in L^1(\mathbb{R}^n)$. Let us consider weak solutions of the initial Cauchy problem (ICP)

$$\left.\begin{aligned} (\beta(v))_t &= \Delta v && \text{in } S_T = \mathbb{R}^n \times (0,T], \\ \beta\,(v(x,0)) &= u_0(x) && x \in \Omega. \end{aligned}\right\} \tag{1.6.7}$$

We have the following existence result for the ICP (1.6.7).

Theorem 1.6.2. *For any $u_0 \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, there exists a solution $v \in L^\infty(S_T) \cap C(S_T)$ of the ICP* (1.6.7).

Proof. We only give a sketch of the proof, leaving the details to the reader. For an increasing sequence of numbers $R_j \uparrow \infty$, let v_j be a solution of the IDP (1.6.2) on $Q_T^j = B_{R_j} \times (0,T]$, with $B_{R_j} = \{x : |x| < R_j\}$. The main estimate is

$$\int_{B_{R_j}} \beta\left(v_j(x,t)\right) dx \le \int_{B_{R_j}} u_0(x)\, dx. \tag{1.6.8}$$

Once this has been established one uses the estimates in Section 1.2, i.e., an priori estimate on the L^∞ of the solution in terms of its L^1 means. □

Chapter 2
The Cauchy problem for slow diffusion

This chapter is concerned with the solvability of the Cauchy problem for the equation

$$\frac{\partial u}{\partial t} = \Delta\varphi(u) \quad \text{on } \mathbb{R}^n \tag{2.0.1}$$

with $\varphi \in \mathcal{S}_a$, corresponding to slow diffusion. We shall provide a complete characterization of non-negative weak solutions of (2.0.1) in terms of their initial condition, showing in particular the results B.1–B.5 stated in the Introduction.

Let us review that for $a \in (0, 1)$, $\mathcal{S}_a$ denotes the class of nonlinearities φ which satisfy the conditions:

(i) $\varphi\colon [0, \infty) \to \mathbb{R}$ is a continuous non-negative function with $\varphi(0) = 0$.

(ii) For all $u > 0$, φ satisfies the growth conditions

$$a \le \frac{u\,\varphi'(u)}{\varphi(u)} \le a^{-1}. \tag{2.0.2}$$

(iii.1) There exist $u_0 > 0$ such that for $u \ge u_0$

$$1 + a \le \frac{u\,\varphi'(u)}{\varphi(u)} \le a^{-1}. \tag{2.0.3}$$

(iii.2) $u_0 = 1$ and $\varphi(1) = 1$.

2.1 Pointwise estimates and existence of initial trace

In this section we shall combine the a priori estimates of Chapter 1, Section 1.2 with the Harnack inequality of Section 1.3 to obtain sharp upper bounds for the size of a solution $u(x, t)$ of (2.0.1), with $\varphi \in \mathcal{S}_a$, on $S_T = \mathbb{R}^n \times (0, T)$ as $t \to 0$ and as $|x| \to \infty$.

For simplicity of the exposition we shall first consider the case of the porous medium equation

$$\frac{\partial u}{\partial t} = \Delta u^m \quad \text{on } S_T = \mathbb{R}^n \times [0, T),\ m > 1 \tag{2.1.1}$$

and at the end we shall explain how one can extend the result to the general class of nonlinearities $\varphi \in \mathcal{S}_a$.

We introduce the following notation: for μ a non-negative measure on $\mathbb{R}^n$ and $\rho \ge 1$ we define

$$|||\mu|||_\rho = \sup_{R>\rho} \frac{\mu(\{|x| < R\})}{R^{n+2/(m-1)}}. \tag{2.1.2}$$

Notice that for $1 \le \rho < r$, $|||\mu|||_r \le |||\mu|||_\rho$ and so

$$|||\mu|||_\infty = \lim_{\rho\to\infty} |||\mu|||_\rho \tag{2.1.3}$$

exists. We shall set $|||\mu||| = |||\mu|||_1$, and for a non-negative function on $\mathbb{R}^n$, $|||f||| = |||f\,dx|||$.

Lemma 2.1.1. *Let u be a non-negative continuous weak solution of (2.1.1). Assume that $0 < \tau < \min(T, 1)$. Then for any $\delta > 0$ there exists constant $C = C(u, \delta, m, n)$ such that for $\tau < t < T - \delta$*

$$u^m(x,t) \le 1 + C\,u(x,t)\,\frac{l^2}{\tau} \tag{2.1.4}$$

where $l = \tau^{1/(n(m-1)+2)}\,(1 + |x|)$.

Proof. The estimate (2.1.4) is trivial if $u(x,t) \le 1$, and so we shall assume that $u(x,t) > 1$.

As a consequence of the Harnack estimate we have that

$$B = \sup_{0<s<T-\delta} |||u(\cdot, s)||| \le C(u, T, \delta).$$

For $(\xi, s) \in \mathbb{R}^n \times (0, \tau^{-1}T)$, set

$$v(\xi,s) = \frac{u(x + l\xi, \tau s)}{\gamma} \quad \text{with } \gamma = \left(\frac{l^2}{\tau}\right)^{1/(m-1)}.$$

Then, for $0 < s < (T-\delta)/\tau$, we have

$$\begin{aligned}
\int_{|\xi|<1} v(\xi,s)\,d\xi &= l^{-n}\gamma^{-1}\int_{|x-\xi|<l} u(\xi,\tau s)\,d\xi \\
&\le l^{-n}\gamma^{-1}\int_{|\xi|<l+|x|} u(\xi,\tau s)\,d\xi \\
&\le C\,B\,l^{-n}\gamma^{-1}(1+|x|)^{n+2/(m-1)} \\
&\le C
\end{aligned}$$

since $l^{-n}\gamma^{-1}(1+|x|)^{n+2/(m-1)} = 1$, by the definition of l, and $l \le 1 + |x|$, if $\tau \le 1$.

By Lemma 1.2.8

$$v(0,1) = \frac{u(x,\tau)}{\gamma} \le C.$$

Hence we conclude the estimate

$$u(x,\tau) \le C\,\gamma = C\left(\frac{l^2}{\tau}\right)^{1/(m-1)}.$$

□

Remark. The estimate (2.1.4) can be rewritten in the following form:

$$u^{m-1}(x,\tau) \le 1 + C\,\tau^{-\frac{n(m-1)}{n(m-1)+2}}\,(1+|x|^2). \tag{2.1.5}$$

The following result generalizes that given in Lemma 1.3.2 in Chapter 1 concerning the existence of a trace.

Theorem 2.1.2. *Let u be a non-negative continuous weak solution of equation* (2.1.1). *Then there exists a measure μ on $\mathbb{R}^n$ with $|||\mu||| < \infty$ such that $u(\cdot,t) \to d\mu$ in $D'(\mathbb{R}^n)$ as $t \downarrow 0$.*

Proof. If $\eta \in C_0^\infty(\mathbb{R}^n)$, it follows from (2.1.5) that

$$\left|\int_{\mathbb{R}^n} [u(x,t) - u(x,\tau)]\,\eta(x)\,dx\right| = \left|\int_\tau^t \int_{\mathbb{R}^n} \Delta\eta\, u^m(x,s)\,dx ds\right|$$
$$\le \int_\tau^t \int_{\mathbb{R}^n} |\Delta\eta|\,(1+|x|^2)\,s^{-\sigma}\,u(x,s)\,dx ds \to 0$$

when $\tau, t \to 0$ since $\sigma < 1$. □

To complete this section we explain how the above results extend to solutions of

$$\frac{\partial u}{\partial t} = \Delta\varphi(u) \quad \text{with } \varphi \in \mathcal{S}_a.$$

(i) In the definition (2.1.2) of $|||\mu|||_\rho$, $R^{2/(m-1)}$ must be replaced by $\Lambda(R^2)$ where $\Lambda(\cdot)$ is the inverse of $\frac{\varphi(u)}{u} : [1,\infty) \to [1,\infty)$.

(ii) For $s > 0$ and $0 < \tau < 1$, notice that the equation

$$R^n\,\Lambda\left(\frac{R^2}{\tau}\right) = (1+s)^n\,\Lambda\big((1+s)^2\big)$$

has a unique solution $R = R(s,\tau)$. Moreover there exists $\delta = \delta(n,a) > 0$ such that

$$R(s,\tau) \le (1+s)\,\tau^\delta.$$

Thus in Lemma 2.1.1 the estimate (2.1.4) must read

$$\varphi\,(u(x,t)) \le 1 + C\,u(x,t)\,\frac{(R(|x|\,,\tau))^2}{\tau}.$$

2.2 Uniqueness of solutions

In this section we shall establish that the solutions in $S_T = \mathbb{R}^n \times [0,T)$ are uniquely determined by their initial trace. We shall present the results in the case of the porous medium equation (2.1.1), referring the reader to [46] for the more general case of equation (2.0.1), with $\varphi \in \mathcal{S}_a$.

We begin with the following preliminary results.

Lemma 2.2.1. *Suppose that u, v are non-negative continuous weak solutions of the equation (2.1.1), such that for any $R > 0$*

$$\lim_{t \downarrow 0} \int_{|x|<R} [v(x,t) - u(x,t)]^+ \, dx = 0$$

where $A^+ = \max\{A, 0\}$. Then $v \le u$ in S_T.

Proof. Define $w = v - u$ and $q = \chi_{\{v>u\}}$ (characteristic function). Let $\eta \in C_0^\infty(\mathbb{R}^n)$, $\eta > 0$.

We claim that for $0 < \tau < t < T$, we have

$$\int_{\mathbb{R}^n} \eta(x) w^+(x,t) \, dx \le \int_{\mathbb{R}^n} \eta(x) w^+(x,\tau) dx + \int_\tau^t \int_{\mathbb{R}^n} \Delta\eta(x) \left[v^m(x,s) - u^m(x,s)\right]^+ \, dxds. \tag{2.2.1}$$

Indeed, by Corollary 1.1.2 in Chapter 1 it suffices to prove (2.2.1) when u and v are smooth functions. By Kato's inequality [98]

$$\Delta(v^m - u^m)^+ \ge q \, \Delta(v^m - u^m)$$

in the distribution sense (d.s.). Since $\partial w^+/\partial t = q \, \partial w/\partial t$ in the d.s. it follows that

$$-\frac{\partial w^+}{\partial t} + \Delta(v^m - u^m)^+ \ge 0$$

in the d.s. Hence

$$\int_\tau^t \int_{\mathbb{R}^n} \left\{-\frac{\partial w^+}{\partial t} + \Delta(v^m - u^m)^+\right\} \eta(x) \, dxds \ge 0$$

and (2.2.1) follows by integration by parts.

Let

$$A = \begin{cases} \frac{v^m - u^m}{v - u} & \text{if } v > u, \\ 0 & \text{otherwise.} \end{cases}$$

We wish to estimate A from above using (2.1.5). In the pure power case that we are presenting here one may easily estimate A using that $A \le m \, v^{m-1}$ and therefore for $0 < t < T - \delta$, $A \le C\,(1 + |x|^2)\, t^{-\sigma}$, by (2.1.5), where $\sigma = \sigma(m,n) \in (0,1)$ and $C = C(u, v, \delta, T)$. However, in the more general case of equation (2.0.1), where $\varphi \in \mathcal{S}_a$ and $\varphi'(u)$ is not necessarily an increasing function one needs to use a different argument.

We shall give in the sequel a proof which immediately generalizes to the more general case. We shall first show that if $v \ge 2$, then for $0 < \delta < T$ and $0 < t < T - \delta$, we have

$$A \le C\,(1 + |x|^2)\, t^{-\sigma} \tag{2.2.2}$$

where $\sigma = \sigma(m, n) \in (0, 1)$ and $C = C(u, v, \delta)$. To prove (2.2.2) consider two cases:

(i) If $0 \le u \le v/2$ and $v \ge 2$ then

$$A \le 2\,\frac{v^m}{v} = 2\,v^{m-1} \le C\,(1 + |x|^2)\,t^{-\sigma}$$

by (2.1.5).

(ii) If $u > v/2$ and $v \ge 2$ then $A \le m z^{m-1}$, for some $z \in (u, v)$. Therefore using (2.1.5) again we obtain (2.2.2).

We next let $\tau \downarrow 0$ in (2.2.1) and use the hypothesis and (2.2.2) we obtain that for $t \in (0, T)$

$$\begin{aligned}\int_{\mathbb{R}^n} w^+(x,t)\,\eta(x)\,dx \le C \int_0^t \int_{|x|\le 2R} &(1+|x|^2)s^{-\sigma} w^+(x,s)\;|\Delta\eta|\,dxds \\ &+ 2\int_0^t \int_{0\le v<2} v^m\;|\Delta\eta(x)|\,dxds.\end{aligned} \tag{2.2.3}$$

For $r > 1$, let

$$M_r(t) = |||w^+(\cdot, t)|||_r = \sup_{R>r} \frac{1}{R^{n+2/(m-1)}} \int_{|x|<R} w^+(x,t)\,dx.$$

Choose $R \ge r$ such that

$$\int_{|x|<R} w^+(x,t)\,dx \ge \frac{1}{2}\,M_r(t)\,R^{n+2/(m-1)}$$

and assume that $\eta \in C_0^\infty(\mathbb{R}^n)$ satisfies

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ on } \{|x| < R\}, \quad \operatorname{supp}\eta \subseteq \{|x| \le 2R\}$$

and

$$|\Delta\eta| \le \frac{C}{R^2}.$$

Substituting in (2.2.3) we obtain

$$\begin{aligned}\frac{1}{2}\,M_r(t)\,R^{n+2/(m-1)} \le C \int_0^t \int_{|x|\le 2R} &(1+|x|^2)s^{-\sigma} w^+(x,s)\;|\Delta\eta|\,dxds \\ &+ 2\int_0^t \int_{0\le v<2} v^m\;|\Delta\eta(x)|\,dxds.\end{aligned}$$

Since $1 < r < R$ it follows that

$$M_r(t) \le C\left\{t\,r^q + \int_0^t s^{-\sigma} M_r(s)\,ds\right\} \tag{2.2.4}$$

where $q = -2 - 2/(m-1)$ for $0 < t < T - \delta$. Defining

$$F(t) = \int_0^t s^{-\sigma} M_r(s)\, ds$$

we have that

$$\frac{d}{dt} F(t) \le C \left\{ \alpha\, t^{-\sigma+1} + t^{-\sigma} F(t) \right\}$$

where $\alpha = r^q$, for $t \in (0, T-\delta)$. Note that in the pure power case we can take $\alpha = 0$, so that the above ODE readily shows that $F(t) \equiv 0$, since $F(0) = 0$ finishing the proof of the lemma. However, for a proof that can be generalized in the case of equation (2.0.1) one needs to continue the argument, as shown next.

It is then easy to see that

$$F(t) \le C'\alpha \quad \text{for } 0 < t < T - \delta$$

where $C' = C'(u, v, \delta, T)$. Hence, by (2.2.4)

$$M_r(t) \le C\, \alpha = C\, r^{-2-\frac{2}{m-1}}.$$

In particular, for $1 < r < R$

$$\int_{|x|<r} w^+(x,t)\, dx \le C\, r^{n-2}, \quad 0 < t < T - \delta. \tag{2.2.5}$$

Now we shall combine (2.2.3) and (2.2.5) to improve (2.2.5). Notice that for $0 < u < v < 2$ we have $v^m - u^m \le C_m\,(v-u)$. Hence, if $R > r$ and η is chosen as before, then by (2.2.3)

$$\begin{aligned}
&\int_{|x|<R} w^+(x,t)\, dx \\
&\quad \le \int_0^t \int_{\mathbb{R}^n} \left[v^m - u^m \right]^+ |\Delta \eta|\; dx dt \\
&\quad \le C\, R^{n+\frac{2}{m-1}} \int_0^t s^{-\sigma} M_r(s)\, ds + C\, R^{-2} \int_0^t \int_{|x|<2R} w^+(x,s)\, dx ds.
\end{aligned}$$

Dividing through by $R^{n+2/(m-1)}$ and using (2.2.5) and the fact that $R > r$ we obtain that

$$M_r(t) \le C \int_0^t s^{-\sigma} M_r(s)\, ds + C\, t\; r^{-2/(m-1)-4}. \tag{2.2.6}$$

By the same method as above, where instead of (2.2.4) we use (2.2.6), we conclude the improved estimate

$$\int_{|x|<r} w^+(x,t) dx \le C r^{n-4} \quad \text{for } r > 1,\, 0 < r < T - \delta. \tag{2.2.7}$$

Let $h(x,t) = \int_0^t \left[v^m(x,s) - u^m(x,s)\right]^+ ds$. Then, arguing as above we find using (2.2.2) that

$$\begin{aligned} r^{-n}\int_{|x|\le r} h(x,t)\,dx &\le C\int_0^t r^{-n}\int_{|x|\le r} w^+(x,s)\,dxds \\ &\quad + C\,r^{2-n}\int_0^t s^{-\sigma}\int_{|x|<r} w^+(x,s)\,dxds \\ &\le C\,tr^{-4} + C\,t^{-\sigma+1}r^{-2}. \end{aligned}$$

However h is subharmonic in x for every $t \in (0,T)$, because for $\eta \in C_0^\infty(\mathbb{R}^n)$, $\eta \ge 0$ we have

$$\int_{\mathbb{R}^n} \eta(x)\,\Delta h(x,t)\,ds \ge \int_0^t\int_{\mathbb{R}^n} \eta(x)\,\frac{\partial w^+}{\partial s}\,dxds = \int_{\mathbb{R}^n} \eta(x) w^+(x,t)\,dx \ge 0.$$

Hence as

$$h(x,t) \le C_n\,r^{-n}\int_{|x-y|<r} h(y,t)\,dy \to 0, \quad r\to\infty,$$

h is identically 0, which easily yields the lemma for the case $n \ge 3$. For $n = 1, 2$ define $u^*(\xi,t) = u(x,t)$ with $\xi = (x,y) \in \mathbb{R}^{n+2}$ and notice that u^* solves $\partial u/\partial t = \Delta u^m$, in $\mathbb{R}^{n+2}$. Therefore using the result for $n \ge 3$ we complete the proof. □

Now we shall prove the general uniqueness result by using the approximation technique in [45].

Theorem 2.2.2. *Let u, v be continuous non-negative weak solutions of equation (2.1.1) and assume that*

$$\lim_{t\downarrow 0}\,[u(\cdot,t) - v(\cdot,t)] = 0 \quad \text{in } D'(\mathbb{R}^n)$$

Then $u \equiv v$.

Proof. Let μ be the common trace (its existence follows from Theorem 2.1.2). Pick $h \in C_0^\infty(\mathbb{R}^n)$ with $0 \le h \le 1$, and denote by $w(x,t,h)$ the solution of the porous medium equation in $\mathbb{R}^n \times (0,\infty)$ with initial trace $h\mu$ and $\sup_{t>0}\int w(x,t,h) < \infty$. The existence result follows from Lemma 1.3.4 and Theorem H.2 (see the Introduction), while the uniqueness follows from Corollary 1.3.3 in Chapter 1. Moreover,

$$\sup_{t>0}\int w(x,t,h)\,dx \le \int h\,d\mu < \infty$$

Claim. $w \le u$.

For $0 < \varepsilon < T/2$, let U_ε be the unique solution in $\mathbb{R}^n \times (0,\infty)$ of the porous medium equation with data $h(x)u(x,\varepsilon)$ at $t = 0$ and

$$\sup_{t>0}\int U_\varepsilon(x,t)dx \le \int h(x)\,u(x,\varepsilon)\,dx \le C_h$$

where C_h does not depend on ε, by the Harnack estimate. For any continuous function η

$$\lim_{\varepsilon\to 0}\int \eta(x)\,h(x)\,u(x,\varepsilon)\,dx = \int \eta(x)\,h(x)\,d\mu.$$

By the compactness result in $L^\infty(\mathbb{R} : L^1(\mathbb{R}^n))$ (Lemma 1.3.4, Chapter 1).

$$\lim_{\varepsilon\to 0} U_\varepsilon(x,t) = w(x,t)$$

uniformly on compact subsets of $\mathbb{R}^n \times (0,\infty)$. The data for U_ε, namely the function $h(x)\,u(x,\varepsilon)$ belongs to $L^1(\mathbb{R}^n)$. Moreover from Lemma 1.3.4 of Chapter 1, we have

$$\lim_{t\downarrow 0} U_\varepsilon(x,t) = h(x)\,u(x,\varepsilon) \quad \text{in } L^1(\mathbb{R}^n).$$

Thus by the continuity of $u(x,t+\varepsilon)$ and Lemma 2.2.1

$$U_\varepsilon(x,t) \le u(x,t+\varepsilon) \quad \text{for } 0 < t < T-\varepsilon.$$

Hence $w \le u$ in $\mathbb{R}^n \times (0,T)$, which proves the claim.

Now if we choose $0 \le h_j \le h_{j+1} \le 1$, $h_j \in C_0^\infty(\mathbb{R}^n)$ with $\lim_{j\to\infty} h_j(x) \equiv 1$ for all $x \in \mathbb{R}^n$, it follows that

$$w(x,t,h_j) \le w(x,t,h_{j+1}) \le u(x,t).$$

Let

$$w_\infty(x,t) = \lim_{j\to\infty} w(x,t,h_j).$$

Since $\{w_j\}$ is locally bounded in $\mathbb{R}^n \times (0,T)$, from H.1 it follows that $\{w_j\}$ is locally equicontinuous and hence w_∞ is a continuous weak solution of the porous medium equation in $\mathbb{R}^n \times (0,T)$. Let λ denote the trace of w_∞. Observe that $h_j\,\mu \le \lambda \le \mu$ for all j. Hence $\lambda = \mu$.

Since $w_\infty \le u$ it follows that

$$\lim_{t\to 0}\int_{|x|<R} |u(x,t) - w_\infty(x,t)|^+\,dx = 0 \quad \text{for all } R > 0.$$

Hence $u = w_\infty$ (Lemma 2.2.1), and similarly $v = w_\infty$, which concludes the proof of the theorem. □

Corollary 2.2.3. *Let u, v be solutions of the porous medium equation* (2.1.1)*. Assume that u, v have traces μ, ν respectively. If $\mu \le \nu$, then $u \le v$ in S_T.*

Proof. Let $h_j \in C_0^\infty(\mathbb{R}^n)$, $0 \le h_j \le h_{j+1} \le 1$, and such that $\lim_{j\to\infty} h_j(x) = 1$ for all $x \in \mathbb{R}^n$.

Let u_j, v_j be the solutions of the porous medium equation in $\mathbb{R}^n \times (0,\infty)$ with initial traces $h_j\,\mu$ and $h_j\,\nu$ respectively. Then $u_j \le v_j$ since by Lemma 1.3.4 in Chapter 1, $u_j = \lim_{\varepsilon\to 0} u_{j,\varepsilon}$, $v_j = \lim_{\varepsilon\to 0} v_{j,\varepsilon}$, where $u_{j,\varepsilon}$, $v_{j,\varepsilon}$ are the solutions with initial data $\eta_\varepsilon * (h_j\cdot\mu)$, $\eta_\varepsilon * (h_j\cdot\nu)$ respectively, where $\eta \in C_0^\infty(\mathbb{R}^n)$, $\int\eta = 1$, $\eta \ge 0$ and $\eta_\varepsilon(x) = \varepsilon^{-n}\,\eta\,(x/\varepsilon)$. By Lemma 2.2.1, $u_{j,\varepsilon} \le v_{j,\varepsilon}$ and our claim follows. Since $u = \lim u_j$ and $v = \lim v_j$ we obtain the result. □

Next we have the following general compactness principle.

Theorem 2.2.4. *Let $\{u_k\}$ be a sequence of non-negative continuous weak solutions of the porous medium equation (2.1.1). Assume that $\sup_k u_k(0,T) < C$. Let μ_k denote the initial trace of u_k, and assume that μ_k converges weakly to a non-negative measure μ on $\mathbb{R}^n$. Then there exists a unique continuous weak solution to the porous medium equation (2.1.1) such that u_k converges to u uniformly on compact sets and u has trace μ.*

Proof. From the Harnack inequality in Section 1.3 of Chapter 1 and the pointwise estimates in Section 2.1 of this chapter, it follows that for each compact set $K \subseteq \mathbb{R}^n \times (0,T)$

$$\sup_k \|u_k\|_{L^\infty(K)} < C(K).$$

From the equicontinuity result Theorem H.1 it follows that there exists w uniform local limit of a subsequence of $\{u_k\}$.

Claim. w has trace μ.

In fact on $[0, T/2]$ we have

$$|||u_k(\cdot,t)||| \le C(u_k(0,T)) \le C.$$

By the pointwise bound (2.1.5)

$$u_k^{m-1}(x,t) \le C\,t^{-\sigma}\,(1+|x|^2), \quad 0<\sigma<1,$$

and hence

$$\int_{\mathbb{R}^n} [u_k(x,t) - u_k(x,s)]\,\eta(x)\,dx \equiv \int_s^t \int_{\mathbb{R}^n} \Delta\eta\; u_k^{m-1}(x,\tau)\,u_k(x,\tau)\,dx d\tau \le C\,t^{1-\sigma}$$

independently of k, which completes the proof of the theorem. □

2.3 Existence and blow up

In this section we shall study the solvability and the maximum time interval of existence $[0,T)$ for the porous medium equation (2.1.1), with $m > 1$, shown in [20], referring the reader to [46] for the general case $\partial u/\partial t = \Delta\varphi(u)$, with $\varphi \in \mathcal{S}_a$.

Lemma 2.3.1. *There exists a number $\delta = \delta(m,n) > 0$ such that if $|||\mu||| \le \delta$, there exists a unique solution u of the porous medium equation (2.1.1) in $\mathbb{R}^n \times (0,1)$, with initial trace μ.*

Proof. Using Lemma 2.1.1 and Theorem 2.2.4 it is enough to show that there exist constants $\delta = \delta(m,n)$ and $C = C(m,n)$, such that if $f \ge 0$, $f \in C_0^\infty(\mathbb{R}^n)$ and $|||f||| < \delta$ and u solves

$$\begin{aligned} u_t &= \Delta u^m \quad \text{in } \mathbb{R}^n \times (0,\infty),\\ u(x,0) &= f(x) \quad x \in \mathbb{R}^n, \end{aligned}$$

then

$$\sup_{0<t<1} \| | u(\cdot,t) \| | \le C.$$

Let $g(t) = \||u(\cdot,t)\||$ and $\eta_R(x) = \eta(x/R)$, where $0 \le \eta \le 1$, $\eta \in C_0^\infty(\mathbb{R}^n)$, $\eta = 1$ for $|x| \le 1$, and $\eta = 0$ for $|x| \ge 2$. Then

$$\begin{aligned}\int_{\mathbb{R}^n} u(x,t)\,\eta_R(x)\,dx \\ = \int_{\mathbb{R}^n} f(x)\,\eta_R(x)\,dx + \int_0^t \int_{\mathbb{R}^n} R^{-2}\,\Delta\eta\Big(\frac{x}{R}\Big)\,u^m(x,\tau)\,dxd\tau.\end{aligned} \tag{2.3.1}$$

For $0 < \tau < 1$, $R > 1$ define $v(x,s) = u(\rho x, s\tau)/\gamma$, where $\rho = 2R$, $\gamma = (\rho^2/\tau)^{1/(m-1)}$. Thus v is again a solution of the porous medium equation. By Theorem 1.2.8 in Chapter 1, we have

$$\sup_{|x|\le 1} v(x,1) \equiv \sup_{|x|\le 2R} \frac{u(x,\tau)}{\gamma} \le C\,\{I^\theta + I^\sigma\}$$

where $I = \sup_{1/2<s<1} \int_{|x|<2} v(x,s)\,dx$.

Define $G(\tau) = \sup\{g(s) : \tau/2 < s < \tau\}$ so that

$$I \le G(\tau)\,\tau^{1/(m-1)}$$

and

$$\sup_{|x|<2R} u(x,\tau) \le C\gamma\,\{\tau^{\theta/(m-1)}(G(\tau))^\theta + \tau^{\sigma/(m-1)}\,(G(\tau))^\sigma\}.$$

Using (2.3.1) we obtain that

$$\begin{aligned}\int_{|x|<R} u(x,t)dx \le \int_{|x|<2R} f(x)dx + \int_0^t \int_{|x|<2R} u(x,\tau)\,\{\tau^{\sigma-1}\,(G(\tau))^{\sigma(m-1)} \\ + \tau^{\theta-1}\,(G(\tau))^{\theta(m-1)}\}\,dxd\tau\end{aligned}$$

and therefore

$$G(t) \le C\,\delta + C\int_0^t \tau^{\sigma-1}(G(\tau))^{\sigma(m-1)+1} + \tau^{\theta-1}\,(G(\tau))^{\theta(m-1)+1}\,d\tau.$$

It is then easy to see that there are M_0 and $\delta_0 > 0$ such that if $0 < \delta < \delta_0$, then

$$G(\tau) \le M_0 \quad \text{for } \tau \in (0,1),$$

finishing the proof. □

Proposition 2.3.2. *Let u be a non-negative continuous weak solution of the porous medium equation (2.1.1). If μ is the initial trace of u then*

$$\||\mu\||_\infty \le C\,T^{-1/(m-1)}. \tag{2.3.2}$$

Proof. By the Harnack inequality

$$\int_{|x|\le R} u(x,0)dx \le C\left\{R^n\left(\frac{R^2}{T}\right)^{1/(m-1)} + T^{n/2}H_m(u(0,T))\right\}$$

Dividing by $R^{n+2/(m-1)}$ and letting $R\uparrow\infty$ we obtain (2.3.2). □

Theorem 2.3.3. *Let μ be a positive measure with $|||\mu|||_\infty < \infty$. Then there exists $T = T(\mu) > 0$ such that μ is the initial trace of a solution u of equation (2.1.1) defined in $S_T = \mathbb{R}^n \times [0,T)$. More precisely there exists $c = c(m,n)$ such that*

$$|||\mu|||_\infty \ge c\,(T(\mu))^{-1/(m-1)}, \tag{2.3.3}$$

where

$$T(\mu) = \sup\{\,T : \mu \textit{ is the trace of a solution } u \textit{ of } (2.1.1)\,\}.$$

Proof. For $0 < \tau < \rho^2$, $\gamma = (\rho^2/\tau)^{1/(m-1)}$, we have that if u is defined in S_T then $v(x,t) = u(\rho x, \tau t)/\gamma$ is defined in $S_{T\tau^{-1}}$. Let λ be a measure defined by

$$\eta(x)\,d\lambda = \rho^{-n}\,\gamma^{-1}\,\eta\Big(\frac{x}{\rho}\Big)\,d\mu$$

so that if μ is the initial trace of u then λ is the initial trace of v.

For $r > 1$

$$\begin{aligned}\lambda(\{|x|<r\}) &= \rho^{-n}\,\gamma^{-1}\,\mu(\{|x|<\rho r\})\\ &\le |||\mu|||_\rho\, r^n\,\gamma^{-1}(\rho r)^{2/(m-1)}\\ &= |||\mu|||_\rho\, r^{n+2/(m-1)}\,\tau^{1/(m-1)}.\end{aligned}$$

Therefore

$$|||\lambda||| \le \tau^{1/(m-1)}\,|||\mu|||_\rho.$$

Suppose $|||\mu|||_\infty < \infty$. We pick τ such that $|||\mu|||_\infty\tau^{1/(m-1)} < \delta$ (δ as in Lemma 2.3.1) and find $\rho > 1$ such that $|||\mu|||_\rho\tau^{1/(m-1)} < \delta$. Therefore $|||\lambda||| < \delta$, and hence v is defined in S_1 with trace λ. □

Corollary 2.3.4. *μ is the trace of a solution u of the porous medium equation $\partial u/\partial t = \Delta u^m$ in $S_\infty = \mathbb{R}^n \times [0,\infty)$ if and only if $|||\mu|||_\infty = 0$.*

2.4 Proof of Pierre's uniqueness result

This section is concerned with the proof of the uniqueness result H.3 stated in the Introduction. This result has been used in the previous sections and will be proven independently of the results in those sections.

Theorem 2.4.1. *Let u, v be continuous non-negative weak solutions of $\partial z/\partial t = \Delta\varphi(z)$ on $\mathbb{R}^n \times (0,\infty)$ with $\varphi \in \mathcal{S}_a$. Assume that*

$$\sup_{t>0} \int [u(x,t) + v(x,t)]\, dx \le C$$

for some $C > 0$ and that $u, v \in L^\infty(\mathbb{R}^n \times [\tau,\infty))$ for each $\tau > 0$. If

$$\lim_{t\downarrow 0} \int [u(x,t) - v(x,t)]\, \eta(x)\, dx = 0$$

for any $\eta \in C_0^\infty(\mathbb{R}^n)$, then $u \equiv v$.

Proof. Define

$$A = \begin{cases} \frac{\varphi(u)-\varphi(v)}{u-v} & u \ne v, \\ 0 & u = v, \end{cases}$$

and let $\varepsilon_k \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ such that $\varepsilon_k > 0$, and

$$\|\varepsilon_k\|_{L^\infty(\mathbb{R}^n)} + \|\varepsilon_k\|_{L^1(\mathbb{R}^n)} \to 0 \quad \text{as } k \to \infty.$$

Also define

$$\alpha_k = \frac{|\varphi(u) - \varphi(v)|}{|u-v| + \varepsilon_k},$$

and

$$A_k = \alpha_k + \varepsilon_k.$$

Notice that A_k is continuous and strictly positive on compact sets.

Fix $\eta \in C_0^\infty(\mathbb{R}^n)$, $\eta \ge 0$, and pick $R > 1$ such that support $\eta \subseteq B_R(0)$.

Step 1. Denote by θ_R the solution of the problem

$$\begin{aligned} \Delta\theta_R &= -\eta \quad \text{in } B_R(0), \\ \theta_R &= 0 \quad \text{on } \partial B_R. \end{aligned}$$

Fix $T > 0$, and define, for any smooth positive function $\alpha\colon \mathbb{R}^{n+1} \to \mathbb{R}$, $\psi = S(\alpha, R)$ as the solution of the heat equation

$$\left.\begin{aligned} \psi_t + \alpha\,\Delta\psi &= 0 && \text{in } B_R \times (0,T], \\ \psi(x,T) &= \theta_R(x) && \text{in } B_R, \\ \psi(x,t) &= 0 && \text{on } \partial B_R \times [0,T]. \end{aligned}\right\} \tag{2.4.1a}$$

Setting $h = \Delta\psi$ we see that

$$\left.\begin{aligned} h_t + \Delta(\alpha h) &= 0 && \text{in } B_R \times (0,T], \\ h(x,T) &= -\eta(x) && \text{in } B_R, \\ h &= 0 && \text{on } \partial B_R \times [0,T]. \end{aligned}\right\} \tag{2.4.1b}$$

By integration by parts (the same argument used in the proof of Theorem 1.1.1, estimate (1.1.5) in Chapter 1) we obtain that

$$\iint_{B_R\times[0,T]} \alpha\,(\Delta\psi)^2 = \iint_{B_R\times[0,T]} \alpha\, h^2 \le \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\eta|^2\, dx. \tag{2.4.2}$$

Also by the Classical maximum principle

$$0 \le \psi \le \|\theta_R\|_\infty .$$

We claim that

$$h(x,t) \le 0. \tag{2.4.3}$$

Indeed, by integration it follows that for $\tau \in (0,T)$

$$\begin{aligned}
&\int_{B_R} h(x,T)\,\tilde\psi(x,T)\,dx - \int_{B_R} h(x,\tau)\,\tilde\psi(x,\tau)\,dx \\
&\quad= \int_\tau^T\int_{B_R} (h(x,t)\,\tilde\psi(x,t))_t\,dxdt = \int_\tau^T\int_{B_R} \left(h_t\,\tilde\psi + h\,\tilde\psi_t\right)dxdt \qquad (2.4.4)\\
&\quad= \int_\tau^T\int_{B_R} (-\Delta(\alpha h)\,\tilde\psi + h\,\tilde\psi_t)\,dxdt = \int_\tau^T\int_{B_R} (\tilde\psi_t - \alpha\,\Delta\tilde\psi)\,h\,dxdt
\end{aligned}$$

for any smooth function $\tilde\psi$ defined in $B_R \times (\tau,T)$ which vanishes on $\partial B_R \times (\tau,T)$.

In particular we choose $\tilde\psi$ to be the solution of the problem

$$\begin{aligned}
\tilde\psi_t - \alpha\Delta\tilde\psi &= 0 \qquad B_R \times (\tau,T),\\
\tilde\psi &= 0 \qquad \text{on } \partial B_R \times (\tau,T),\\
\tilde\psi(x,\tau) &= w(x) \quad x \in B_R,
\end{aligned}$$

where $w \in C_0^\infty(B_R(0))$, $w \ge 0$.

From (2.4.4) it follows that

$$\int_{B_R} h(x,T)\,\tilde\psi(x,T)\,dx = \int_{B_R} h(x,\tau)\,w(x)\,dx.$$

Since $h(x,T) = -\eta(x) \le 0$, and by the maximum principle $\tilde\psi \ge 0$, we conclude that

$$\int_{B_R} h(x,\tau)w(x)dx \le 0.$$

Therefore $h(x,\tau) \le 0$, which proves the claim (2.4.3).

From (2.4.3) it follows that

$$\begin{aligned}
\frac{d}{dt}\int_{B_R} |h(x,t)|\,dx &= -\int_{B_R} h_t(x,t)\,dx\\
&= \int_{B_R} \Delta(\alpha h)\,dx = \int_{\partial B_R} \frac{\partial(\alpha h)}{\partial n}\,d\sigma \ge 0
\end{aligned}$$

since $h(x,t) \le 0$ on $B_R \times (0,T)$, and vanishes on the boundary. Hence,

$$\int_{B_R} |h(x,t)|\, dx$$

is monotone increasing, and

$$\int_{B_R} |h(x,t)|\, dx \le \int_{B_R} \eta(x) dx \tag{2.4.5}$$

for any $t \in [0,T]$. Also,

$$\frac{\partial \psi}{\partial t} = -\alpha\, \Delta\psi = -\alpha h > 0 \tag{2.4.6}$$

i.e. $\psi(\,\cdot, t)$ is an increasing function. Therefore for any $t \in (0,T)$

$$0 \le \psi(x,t) \le \theta_R(x). \tag{2.4.7}$$

By rescaling (using that θ_R is a Green's potential)

$$\left\| \frac{\partial \psi}{\partial n} \right\|_{L^\infty(\partial B_R)} \le C(\eta)\, R^{1-n}. \tag{2.4.8}$$

Step 2. Let $b = u - v$, where u, v are continuous weak solutions of $\partial z/\partial t = \Delta\varphi(z)$. Then

$$\begin{aligned} E &:= \int_{B_R} b(x,T)\, \theta_R(x)\, dx - \int_{B_R} b(x,\tau)\, \psi(x,\tau)\, dx \\ &= \int_{B_R} \int_\tau^T \frac{\partial}{\partial t}\, (b(x,t)\, \psi(x,t))\; dt dx \\ &= \int_{B_R} \int_\tau^T [\, \Delta(\varphi(u) - \varphi(v))\, \psi - b\, \alpha\, \Delta\psi\,]\, dt dx \\ &= \int_{B_R} \int_\tau^T b\,(A-\alpha)\Delta\psi\; dt dx - \int_{\partial B_R} \int_\tau^T \frac{\partial\psi}{\partial n}\, [\, \varphi(u) - \varphi(v)\,]\; dt d\sigma. \end{aligned} \tag{2.4.9}$$

Denoting by H the last term above, it follows from (2.4.8) that

$$|H| \le C(\eta)\, R^{1-n} \int_\tau^T \int_{\partial B_R} [\, \varphi(u) + \varphi(v)\,]\; d\sigma dt.$$

Using that there exists $\delta \in (0,1)$ such that

$$\varphi(u) \le C\,(u^\delta + u^{1/\delta})$$

and the hypothesis that $u, v \in L^\infty(\mathbb{R}^n \times [\tau, T])$ we get

$$|H| \leq C_{\delta,\tau,T}(\eta) \left\{ R^{1-n} \int_\tau^T \int_{\partial B_R} (u+v)\, d\sigma dt \right. \tag{2.4.10}$$

$$\left. + \left(R^{1-n} \int_\tau^T \int_{\partial B_R} (u+v)\, d\sigma dt \right)^\delta \right\}. \tag{2.4.11}$$

Step 3. We would like to solve (2.4.1a), (2.4.1b) with $\alpha = A$. However we cannot do this. So we use an approximation argument. Since A_k are continuous and strictly positive on compact sets we can find $a_\nu \in C^\infty$ $a_\nu > 0$, with $a_\nu \to A_k$ uniformly on compacts. Let ψ_ν, h_ν be the solutions of the (BVP) (2.4.1a), (2.4.1b) respectively with $\alpha = a_\nu$. Let $\{\tau_j\}$ be a sequence such that $T = \tau_0 > \tau_1 > \cdots \tau_j > \cdots, \tau_j \to 0$ as $j \to \infty$. We consider the sequence $\{-h_\nu(x, \tau_j)\, dx\}$. From (2.4.5) we have that passing to a subsequence, there exists $\lambda_j^{(k,R)}$, the weak limit of $\{-h_\nu(x,\tau_j)dx\}_{\nu=0}^\infty$, as $\nu \to \infty$.

Let $\psi_j^{(k,R)}$ be the Green's potential of $\lambda_j^{(k,R)}$. We claim that

$$\psi_j^{(k,R)} = \lim_{\nu_s \to \infty} \psi_{\nu_s}(x, \tau_j) \tag{2.4.12}$$

for all x where $\{\psi_{\nu_s}\}$ is a subsequence of $\{\psi_\nu\}$.

Since $\psi_\nu(\cdot, \tau_j) = G_R(-h_\nu(\cdot, \tau_j))$, from (2.4.5) it suffices to show that if $\mu_\nu \to \mu$ weakly then for some subsequence $\{\mu_{\nu_s}\}$, $G_R\mu_{\nu_s}(x) \to G_R\mu(x)$ for all $x \in B_R$. For any $\lambda > 0$ we have

$$\begin{aligned}
&\left|\{x : |(G\mu_\nu - G\mu)(x)| > \lambda\}\right| \\
&\quad \leq \left|\{x : |(G^\varepsilon\mu_\nu - G^\varepsilon\mu)(x)| > \lambda/2\}\right| + \left|\{x : |(G_\varepsilon\mu_\nu - G_\varepsilon\mu)(x)| > \lambda/2\}\right| \\
&\quad = |A_\nu| + |B_\nu|
\end{aligned}$$

where $G_\varepsilon(x,y) = \chi_{\{(x,y):|x-y|<\varepsilon\}} G_{(x,y)}$ and $G^\varepsilon(x,y) = G(x,y) - G_\varepsilon(x,y)$. By Chebishev's inequality

$$|B_\nu| \leq \frac{C}{\lambda} \int_{B_R} \int_{|x-y|<\varepsilon} G_\varepsilon(x,y)\ |(d\mu_\nu - d\mu)(y)|\ dx = 0(\varepsilon)$$

since $\|d\mu_\nu - d\mu\| \leq M$ for any ν. On the other hand $G^\varepsilon \in C^\infty$ therefore $G^\varepsilon\mu_\nu(x) \to G^\varepsilon\mu(x)$ a.e. in x as $\upsilon \uparrow \infty$. Remember that G_R the Green's potential takes finite measures into the space $L_\infty^{n/(n-2)}(B_R(0))$ (i.e. $L^{n/(n-2)}(B_R(0))$-weak space). Combining this with the Egorov's theorem we obtain that $G\mu_\nu \to G\mu$ in measure as $\nu \uparrow \infty$. Therefore for some subsequence $\{\mu_{\nu_s}\}$, we have $G\mu_{\nu_s} \to G\mu$ a.e. in x as $\nu_s \uparrow \infty$. Hence $\psi_j^{(k,R)}(x) = \lim_{\nu_s \to \infty} \psi_{\nu_s}(x, \tau_j)$ a.e. in x.

But since $\psi_j^{(k,R)}$, $\psi_\nu(\cdot, \tau_j)$ are Green's potentials, every point is a Lebesgue point, i.e.,

$$\psi_j^{(k,R)}(x) = \lim_{r \downarrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} \psi_j^{(x,R)}(y) dy$$

yielding the claim (2.4.12).

On the other hand, from (2.4.6) it follows that

$$0 \le \psi_{j+1}^{(k,R)}(x) \le \psi_j^{(k,R)}(x) \le \theta_R(x) \quad \text{for all } x \in B_R. \tag{2.4.13}$$

Also by (2.4.5)

$$\int_{B_R} d\lambda_j^{(k,R)} \le \int \eta(x)\,dx. \tag{2.4.14}$$

Setting $\tau = \tau_j$ in (2.4.9) we obtain

$$\left| \int_{B_R} b(x,T)\,\theta_R(x)\,dx - \int_{B_R} b(x,\tau_j)\,\psi_j^{(k,R)}\,dx \right| \le E_j^{(k,R)}$$

where by (2.4.9), (2.4.10) and (2.4.2)

$$\begin{aligned} \left|E_j^{(k,R)}\right| \le C(\eta)\Big[\int_{\tau_j}^T \int_{B_R} A_k^{-1} b^2 (A-A_k)^2 dx dt\Big]^{1/2} \\ + R^{1-n}\int_{\tau_j}^T \int_{\partial B_R} (u+v)\,d\sigma dt \\ + \left(R^{1-n}\int_{\tau_j}^T \int_{\partial B_R} (u+v)\,d\sigma dt\right)^\delta. \end{aligned} \tag{2.4.15}$$

But

$$A_k^{-1} b^2 (A-A_k)^2 \le C\,\varepsilon_k \quad \text{on } B_R \times [\tau_j, T]$$

and thus the first term on the right hand side in (2.4.15) tends to zero as $k \uparrow \infty$ uniformly in R. Since

$$\sup_{t>0} \int_{\mathbb{R}^n} (u+v)\,dx < \infty$$

for some $\{R_l\}$

$$R_l^{1-n} \int_{\partial B_{R_l}} \int_{\tau_j}^T (u+v)\,dt d\sigma \to 0 \quad \text{as } R_l \uparrow \infty.$$

Since

$$\int d\lambda_j^{(k,R)} \le \int \eta(x)\,dx$$

by (2.4.14), there exists a subsequence $d\lambda_j^{(k_s,R_s)}$ such that $\lambda_j^{(k_s,R_s)} \to \lambda_j$ weakly as $s \to \infty$. Let $\psi_j = N\lambda_j =$ Newton potential of λ_j. Since $\psi_j^{(k_s,R_s)} = G\lambda_j^{(k_s,R_s)}$ arguing as above we obtain that

$$\psi_j^{(k_s,R_s)} \to \psi_j \quad \text{a.e. in } x \ \text{(passing to subsequence).}$$

Thus we have

$$0 \le \psi_{j+1} \le \psi_j(x) \le \theta(x) = N\eta(x) \quad \text{for all } x \tag{2.4.16}$$

and

$$\int_{\mathbb{R}^n} d\lambda_j \le \int_{\mathbb{R}^n} \eta(x)\,dx. \tag{2.4.17}$$

From the Dominated Convergence Theorem and (2.4.15) it follows that

$$\int_{\mathbb{R}^n} b(x,T)\,\theta(x)dx = \int_{\mathbb{R}^n} b(x,\tau_j)\,\psi_j(x)\,dx. \tag{2.4.18}$$

Define $\psi_\infty(x) = \lim_{j\to\infty} \psi_j(x)$ (the convergence follows by (2.4.16)).

We claim that

$$\{\lambda_j\} \text{ converges weakly to a measure } \lambda_\infty. \tag{2.4.19}$$

By (2.4.17) it suffices to show that if λ is the weak limit of a subsequence $\{\lambda_{j_s}\}$ of $\{\lambda_j\}$, then $\lambda = \lambda_\infty$, where $\lambda_\infty = -\Delta\psi_\infty$ in D'. Indeed, if

$$\int_{\mathbb{R}^n} \lambda_{j_s}\,\phi \to \int_{\mathbb{R}^n} \lambda\,\phi \quad \text{as } j_s \to \infty$$

then

$$-\int_{\mathbb{R}^n} \psi_{j_s}\Delta\,\phi \to \int \lambda\,\phi$$

for any $\phi \in C_0^\infty$. But by the Dominated Convergence Theorem

$$\int_{\mathbb{R}^n} \psi_{j_s}\,\Delta\phi \to -\int_{\mathbb{R}^n} \psi_\infty\,\Delta\phi = \int \lambda_\infty\,\phi.$$

Hence $\lambda = \lambda_\infty$. Since $-\int_{\mathbb{R}^n} \psi_\infty\,\Delta\phi = \int_{\mathbb{R}^n} \lambda_\infty\,\phi$ we obtain that $\psi_\infty = N\lambda_\infty$ a.e. in x.

We next claim that

$$\psi_\infty \ge N\lambda_\infty \quad \text{everywhere}. \tag{2.4.20}$$

(Notice that we do not know if the measures are absolutely continuous therefore sets of measure zero play an important role). To prove the claim, for $\varepsilon > 0$ define $K_\varepsilon(x) = \min\{k(x), c_n\varepsilon^{2-n}\}$, with $k(x) = c_n\,|x|^{2-n}$ and $N_\varepsilon f = k_\varepsilon * f$. Since $\lambda_j \to \lambda_\infty$ weakly, $N_\varepsilon\lambda_j \to N_\varepsilon\lambda_\infty$ everywhere as $j \to \infty$. Then since

$$N_\varepsilon\lambda_j \le N\lambda_j = \psi_j$$

we have that

$$N_\varepsilon\lambda_\infty \le \psi_\infty.$$

Hence $N\lambda_\infty \le \psi_\infty$, which proves the claim.

We next observe that the solutions u and v have an initial trace, as it follows from Lemma 1.3.2 in Chapter 1. By assumption u, v have the same trace μ.

Our next goal is to show that $Nu(x, \tau_j) \to N\mu$. To this end we shall first prove that $w(x,t) = Nu(x,t)$ satisfies $\partial w/\partial t = -\varphi(u) \le 0$, which will allow us to take limits at $t \to 0$. Indeed, fix $\tau \in (0, T)$ and define

$$e(x, \tau, T) = w(x, T) - w(x, \tau) + \int_\tau^T \varphi(u(x,t))\,dt.$$

Then $\Delta_x e \equiv 0$ in D'. Also $\int_{|x|<R} w(x,t)dx \le C\,R^{n-2}$ for $t \in (\tau, T)$ since $w(x,t) = Nu(x,t)$ and $u \in L^\infty(\mathbb{R}^n \times [\tau, T])$. Using that $\varphi \in \mathcal{S}_a$ we have that for $u \le 1$, $\varphi(u) \le u^a$, with $a < 1$. For any $p > 1$, we have

$$\left(\int_{\mathbb{R}^n}\left(\int_\tau^T \varphi(u(x,t))dt\right)^p dx\right)^{1/p} \le \int_\tau^T \left(\int_{\mathbb{R}^n} (\varphi(u))^p\,dx\right)^{1/p} dt. \tag{2.4.21}$$

But if $p = 1/a$

$$\begin{aligned}\int_{\mathbb{R}^n} (\varphi(u))^p\,dx &= \int_{u\ge 1} (\varphi(u))^p\,dx + \int_{u<1} (\varphi(u))^p\,dx\\ &\le M(\tau)\,|\{x : u \ge 1\}| + \int_{u<1} u dx < \infty\end{aligned}$$

where

$$M(\tau) := \sup_{t\in(\tau,T)} (\varphi\,(u(x,t)))^p < \infty$$

by our assumptions. Then

$$\left[\int_{\mathbb{R}^n}\left(\int_\tau^T \varphi\,(u(x,t))\,dt\right)^p dx\right]^{1/p} < \infty$$

i.e. $\int_\tau^T \varphi(u)(\,\cdot, t)\,dt \in L^p(\mathbb{R}^n)$ for some $p > 1$, and thus

$$\int_{|x|<R} |e(x, \tau, T)|\,dx \le C\,R^{n-\delta} \quad \text{for some } \delta > 0.$$

By the Mean Value Theorem for harmonic functions we can conclude that $e \equiv 0$. Hence

$$w(x, \tau) = w(x, T) + \int_\tau^T \varphi\,(u(x,t))\,dt$$

which implies that $w(x, \tau)$ is differentiable in τ and

$$\frac{\partial w}{\partial t} = -\varphi(u) \le 0.$$

For the sequence $\tau_j \downarrow 0$, it follows from the above that $w(\,\cdot, \tau_j)$ is an increasing sequence and thus its limit $F(x)$ is a superharmonic function. Hence, $F = N\mu$, for some $\mu \ge 0$ everywhere (passing to a subsequence we can assume that μ is the weak limit of $u(\,\cdot, \tau_j)$ as $\tau_j \to 0$).

We finally claim that

$$b \equiv 0. \tag{2.4.22}$$

Indeed, pick $j \geq k$, then

$$\int_{\mathbb{R}^n} u(x, \tau_j)\, \psi_j(x)\, dx = \int_{\mathbb{R}^n} u(x, \tau_j)\, N\lambda_j(x)\, dx = \int_{\mathbb{R}^n} w(x, \tau_j)\, d\lambda_j$$

and

$$\begin{aligned}\liminf_{j\to\infty} \int_{\mathbb{R}^n} u(x, \tau_j)\, \psi_j(x)\, dx &\geq \int_{\mathbb{R}^n} u(x, \tau_k)\, \psi_\infty(x)\, dx \\ &\geq \int u(x, \tau_k)\, N\lambda_\infty(x)\, dx \\ &= \int_{\mathbb{R}^n} w(x, \tau_k)\, d\lambda_\infty\end{aligned}$$

where

$$\int_{\mathbb{R}^n} w(x, \tau_k)\, d\lambda_\infty \to \int F(x) d\lambda_\infty \equiv \int N\mu\, d\lambda_\infty$$

as $k \uparrow \infty$.

For $s > 0$ let $v_s(x, t) = v(x, t + s)$, $b_s = u - v_s$ and $w_s = Nv_s$. Then

$$0 \leq w_s \leq C_s \quad \text{on } \mathbb{R}^n \times [0, \infty)$$

and by the Dominated Convergence Theorem

$$\begin{aligned}\limsup_{j\to\infty} \int v_s(x, \tau_j)\, \psi_j(x)\, dx &\leq \limsup_{j\to\infty} \int w_s(x, 0)\, d\lambda_j \\ &= \limsup_{j\to\infty} \int v(x, s)\, \psi_j(x)\, dx \\ &= \int v(x, s)\, \psi_\infty(x)\, dx.\end{aligned}$$

Hence

$$\liminf_{j\to\infty} \int b_s(x, \tau_j)\, \psi_j(x)\, dx \geq \int [N\mu - w_s(x, 0)]\, d\lambda_\infty \geq 0$$

consequently from (2.4.18) (with τ instead of T)

$$\int b_s(x, \tau)\theta(x) dx \geq 0.$$

Thus taking the limit when $s \downarrow 0$ we obtain

$$\int b(x, \tau)\, \theta(x)\, dx \geq 0.$$

Interchanging u and v we have that

$$\int b(x,\tau)\,\theta(x)dx \le 0.$$

Therefore

$$\int b(x,\tau)\,\theta(x)dx = 0 \quad \text{for all } \theta \in C_0^\infty$$

which implies that $b(x,\tau) = 0$ for any x and any τ. □

2.5 Further results

In this section we shall give a brief summary of further known results on the behavior of solutions of the Cauchy problem for equation (2.0.1) in the slow diffusion case, which were not covered in detail in Sections 2.1–2.4, as they are not closely related to the main objective of this book. Most of the results which will be discussed are concerned with the pure power case $u_t = \Delta u^m$, $m > 1$. This equation has been extensively studied in the literature. We apologize for not mentioning all the known results. We refer the reader to the survey articles [111] and [125] for a collection of known results.

1. The variable coefficient porous medium equation. We shall give here a brief summary on how one can extend the results of this chapter to non-negative weak solutions of the variable coefficient generalized porous medium equations

$$\frac{\partial u}{\partial t} = \Delta\varphi(x,t,u), \qquad (x,t) \in \mathbb{R}^n \times (0,T) \tag{2.5.1}$$

for nonlinearities φ which belong to the class $\mathcal{S}_a$ uniformly in (x,t), $x \in \mathbb{R}^n$, $t > 0$, with $\varphi(0,0,1) = 1$. Because of the (x,t) dependence, we impose the following extra assumptions on φ; namely that there exist $\lambda \in (0,1)$ such that

(i) for every $x, x' \in \mathbb{R}^n$, $t, t' > 0$

$$\lambda \le \frac{\varphi(x,t,u)}{\varphi(x',t',u)} \le \lambda^{-1}; \tag{2.5.2}$$

(ii) for every $x \in \mathbb{R}^n$ and $t > 0$, the function $\varphi(x,\cdot,u)$ is differentiable in t and

$$\lambda \le \frac{\varphi_t(x,t,u)}{\varphi(x',t',u)} \le \lambda^{-1} \tag{2.5.3}$$

for all $u \ge 1$.

We denote by $\mathcal{S}_{\alpha,\lambda}$ the class of such nonlinearities. The complete characterization of non-negative continuous weak solutions of the Cauchy problem for (2.5.1), with φ satisfying the conditions described above, is given in [51].

The existence of solutions is obtained in a similar manner as in the constant coefficient case of $\varphi = \varphi(u)$ studied in Section 2.3 of this chapter.

To show uniqueness and existence of initial trace, one needs to establish a Harnack inequality, similar to Theorem 1.3.7 of Chapter 1. One uses compactness arguments, based on the fact that a uniformly bounded sequence $\{u_k\}$ of solutions of $\partial u_k/\partial t = \Delta\varphi_k(x, t, u_k)$, with $\varphi_k \in \mathcal{S}_{a,\lambda}$, is equicontinuous. The difference between the variable coefficient case $\varphi = \varphi(x, t, u)$ and the case of $\varphi = \varphi(u)$ is that the class of nonlinearities $\mathcal{S}_{a,\lambda}$ is not compact because of the $(x, t)-$dependence. A subtle argument needs to be used in order to establish certain weak compactness of the class $\mathcal{S}_{a,\lambda}$.

To establish uniqueness of solutions, one combines the Harnack inequality and techniques from Potential Theory. To overcome certain technical difficulties arising from the time dependence of the coefficients, one establishes the analogue of the Aronson–Bénilan inequality, namely the inequality

$$\frac{\partial u}{\partial t} \geq -\frac{K}{t}\, u(x, t)$$

with K a constant depending only on data a, λ and the dimension n. This inequality was shown for the pure power case in [7]. Its proof in the case of equation (2.5.1) is substantially harder and is given in [51]. We refer the reader to this paper for all the details.

2. Changing sign solutions. We shall comment in this section on possible extensions of the results in this chapter to the case of changing sign solutions of the porous medium equation

$$u_t = \operatorname{div}(|u|^{m-1}\,\nabla u) \quad \text{on } \mathbb{R}^n \times (0, T) \tag{2.5.4}$$

for $m > 1$.

The Cauchy problem for changing sign solutions of (2.5.4) was studied by Bénilan, Crandall and Pierre in [20] where the following existence and uniqueness result was shown.

Theorem 2.5.1 ([20]). *Assume that $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and satisfies the growth assumption*

$$|u_0(x)| = 0(|x|^{2/(m-1)}) \quad \text{as } |x| \to \infty. \tag{2.5.5}$$

Then, there exists a unique $u \in C([0, T); L^1_{\mathrm{loc}}(\mathbb{R}^n))$ which solves (2.5.4) in the distributional sense, takes on the initial data u_0 and satisfies the growth estimate

$$|u(x, t)| = 0(|x|^{2/(m-1)}) \quad \text{as } |x| \to \infty. \tag{2.5.6}$$

locally uniformly in $0 \leq t < T$. Moreover, the existence is global (i.e., $T = \infty$) if $|u_0(x)| = o(|x|^{2/(m-1)})$ as $|x| \to \infty$.

The above theorem settles the questions of existence and uniqueness of changing sign solutions of (2.5.4) in the class $u \in C([0,T); L^1_{\text{loc}}(\mathbb{R}^n))$ and with initial data which satisfies the growth condition (2.5.5). However, the corresponding theory for weak solutions with initial data a measure has not been developed. In particular the uniqueness of changing sign solutions with initial data being a measure has been a long-standing open problem.

In [123], J. L. Vazquez constructed a non-trivial solution u of equation (2.5.4) in dimension $n = 1$ with zero initial data. The solution u has the self-similar form $u(x,t) = t^{-\alpha}\, U(x\, t^{-\beta})$, with $\beta < 0$ and $\alpha = (1-2\beta)/(m-1) > 0$, and satisfies

$$\lim_{t \to 0} u(x,t) = 0$$

uniformly in x on compact subsets of $\mathbb{R}$. However, u becomes oscillatory as $|x| \to \infty$ and does not satisfy the growth condition (2.5.6) locally uniformly in $0 \le t < T$.

One basic question remains open: Let u and v be two changing sign continuous weak solutions of equation (2.5.4) on $\mathbb{R}^n \times (0,\infty)$ such that $u, v \in L^\infty(\mathbb{R}^n \times (\tau,\infty))$, for each $\tau > 0$ and

$$\sup_{t>0} \int_{\mathbb{R}^n} |u(x,t)| + |v(x,t)|\, dx < \infty.$$

If

$$\lim_{t \downarrow 0} \int_{\mathbb{R}^n} [u(x,t) - v(x,t)]\, \eta(x)\, dx = 0$$

for all $\eta \in C_0^\infty(\mathbb{R}^n)$, is $u = v$?

3. The regularity of the free boundary. We consider in this section the Cauchy problem for the porous medium equation

$$\left.\begin{aligned} u_t &= \Delta u^m && \text{in } \mathbb{R}^n \times [0,\infty), \\ u(x,0) &= u_0 && \text{on } \mathbb{R}^n, \end{aligned}\right\} \tag{2.5.7}$$

in the range of exponents $m > 1$, with initial data u_0 non-negative, integrable and *compactly supported.*

The function u represents the density of an idea gas through a porous medium, while $p = m\, u^{m-1}$ represents the pressure of the gas. The function p satisfies the equation

$$p_t = p\, \Delta p + r(m)\, |Dp|^2 \tag{2.5.8}$$

with $r(m) = 1/(m-1)$. When $u = 0$, then $p = 0$ and both of the above equations become degenerate. This degeneracy results into the interesting phenomenon of the *finite speed of propagation*: If the initial data u_0 is compactly supported in $\mathbb{R}^n$, the solution $u(\cdot,t)$ will remain compactly supported for all time t. Hence the boundary of the support of $u(\cdot,t)$, namely the surface $\Gamma(t) = \partial\{u(\cdot,t) > 0\}$ behaves like a free boundary propagating with finite speed. Since the equation becomes degenerate at the free boundary where $u = p = 0$, the solutions u and p are not expected to be smooth near the boundary of its support. The optimal regularity for the density u has been shown to be Hölder continuous and solutions are understood in the distributional sense. On the other hand, the physical interpretation of the equation indicates that, under ideal conditions, the free boundary should be a smooth surface and the pressure p a smooth function up to the interface. However, this is not always the case as advancing free boundaries may hit each other after a short time creating singularities.

In this section we shall present a short summary of the main results concerning the regularity of the free boundary

$$\Gamma = \partial\, \overline{\mathrm{supp}}\, u$$

under rather general initial conditions.

Caffarelli and Friedman [29], [28] showed that the interface can be always described by a Hölder continuous function $t = S(x)$, $x \notin \mathrm{supp}\, u_0$, for any initial data.

In the one dimensional case the free boundary regularity is completely understood. It has been shown in [100] and [29], that if the support of the initial data is an interval, the free boundary consists out of two Lipschitz continuous curves $x = \zeta_i(t)$, with ζ_1 decreasing and ζ_2 increasing. Moreover, there exist waiting times t_i^* so that $\zeta_i(t)$ are constant for $t \le t_i^*$ and each ζ_i is C^1 for $t > t_i^*$. However at $t = t_i^*$, $\zeta_i(t)$ may have a corner, so Lipschitz is the optimal regularity [11]. Aronson and Vazquez [15] and independently Höllig and Kreis [83] showed that for $t > t_i^*$ the curves ζ_i are smooth. Moreover, it was shown in [15] that the pressure p is smooth up to the interface for $t > \max(t_1^*, t_2^*)$ and that it becomes concave for t sufficiently large. Bénilan and Vazquez [19] showed that if $(p_0)_{xx} \le -C < 0$, then the concavity of the pressure p is preserved under the flow and consequently the interfaces $x = \zeta_i(t)$ are concave curves. Angenent [1] showed that for $t > t_i^*$ the curves ζ_i are real analytic.

In dimensions $n \ge 2$ the regularity of the free boundary poses a much harder question. The non-degeneracy of the initial pressure, namely the condition

$$|Dp_0| + p_0 \ge \mu > 0 \quad \text{on } \{\, p_0 > 0\,\} \tag{2.5.9}$$

is crucial for regularity. Condition (2.5.9) ensures that the free boundary will start to move at each point for $t > 0$. Caffarelli, Vazquez and Wolanski [32] established the Lipschitz regularity of the pressure p and the free boundary Γ, away from the initial support, provided that the initial pressure p_0 satisfies (2.5.9) and some extra assumptions. In particular, it was shown in [32] that the free boundary is Lipschitz continuous for $t > T_0$, with T_0 sufficiently large. This result was later improved by Caffarelli and Wolanski in [33], where it was shown the free boundary Γ is $C^{1,\alpha}$, for some $\alpha > 0$. The C^∞-regularity of the pressure p up to the free boundary and the free boundary Γ, for large times $t > T_0$ was recently shown by Koch [98].

The short time C^∞ regularity of the pressure p up to the free boundary and the free boundary Γ under the non-degeneracy condition (2.5.9) was established independently by Daskalopoulos and Hamilton [57], [58], Koch [98] and Meirmanov, Pukhnachov, Shmarev [108]. The main tools in [57], [58] and [98] are local Schauder estimates and local $W^{2,p}$ estimates respectively, which are scaled according to a singular metric, which is appropriate for the degenerate problem. The short time analyticity of the free boundary is shown in [98]. The results in [57], [58] and [98] require that the initial data p_0 is at least of class $C^{1,\alpha}(\Omega)$, for some $\alpha > 0$, with Ω denoting the closure of the initial support. The short time C^∞ regularity of the free boundary with Lipschitz continuous initial data satisfying (2.5.9) remains an open question.

In [60], Daskalopoulos, Hamilton and Lee studied the convexity properties of the porous medium equation in connection to the all-time regularity of free boundary. They showed that if the initial data p_0 is root-concave and satisfies (2.5.9), then the root-concavity of p is preserved for $t > 0$. As a consequence, it was shown in [60], that the free boundary remains smooth for all times. It is a rather interesting observation that in space dimensions $n \geq 2$ it is the concavity of $\sqrt{p}$ and not the concavity of p that is preserved under the flow. However, one may ask: does p become concave for large times? The answer is indeed affirmative as was shown by Lee and Vazquez in [104].

The free boundary regularity for solutions to the slow diffusion equation $u_t = \Delta\varphi(u)$, under certain conditions on the nonlinearity φ which generalize the porous medium equation has been studied by Daskalopoulos and Rhee in [61]. We refer the reader to their paper for detailed statements of the results and proofs.

In the so called focusing problem one studies weak solutions to the porous medium equation (2.5.7) whose initial support $\mathrm{supp}\{u_0 > 0\}$ lies in the exterior of some compact set, for example a ball. At a finite time T the gas will fill all the initially empty region. In [13], Aronson and Graveleau constructed an one-parameter family $\{g_c\}$ of self-similar solutions to the radially symmetric focusing problem. These solutions are an example of self-similar solutions of the second kind, in which the self-similarity variable cannot be determined a priori from dimensional considerations. The Aronson–Graveleau self-similar solutions show that in more than one space dimensions the gradient of the pressure p is infinite at the center of the empty region R at the focusing time T. This means that the pressure p of solutions to the porous medium equation is in general not Lipschitz continuous. The optimal Hölder exponent of the density u is unknown.

Angenent and Aronson [2], [3] showed that the focusing behavior of radially symmetric solutions is determined by one member of the family $\{g_c\}$. In other words the family $\{g_c\}$ determines the local intermediate asymptotics of the focusing process. In [4] Angenent and Aronson construct non-radially symmetric self-similar solutions to the porous medium equation, by showing that the family of radial self-similar solutions $\{g_c\}$ undergoes a sequence of breaking bifurcations as the exponent m decreases from $m = \infty$ to $m = 1$.

4. Time asymptotic behaviour of solutions. In this section we shall present a brief summary of the results on the asymptotic in time behaviour of solutions of the Cauchy problem

$$\left.\begin{aligned} u_t = \Delta u^m \quad & \text{in } \mathbb{R}^n \times [0,T), \\ u(x,0) = u_0 \quad & \text{on } \mathbb{R}^n, \end{aligned}\right\} \tag{2.5.10}$$

for the porous medium equation $m > 1$. Let u be a non-negative weak solution of (2.5.10) with non-negative and integrable initial data

$$u_0 \in L^1(\mathbb{R}^n) \quad u_0 \geq 0.$$

Since $u_0 \in L^1(\mathbb{R}^n)$, the solution u will exists up to time $T = \infty$ (Theorem 2.3.3). The large time behaviour of solutions in this class is described by the one parameter family of source type self-similar solutions

$$U(x,t;C) = t^{-\alpha} F(xt^{-\beta};C) \tag{2.5.11}$$

with parameter $C > 0$ [17], [136], called Barenblatt solutions. The profile F is given by the explicit formula

$$F(\eta) = (C - k\,\eta^2)_+^{\frac{1}{m-1}}, \quad k = \beta(m-1)/2m \tag{2.5.12}$$

and the constants α, β are given by

$$\alpha = \frac{n}{n(m-1)+2}, \quad \beta = \frac{1}{n(m-1)+2}. \tag{2.5.13}$$

The solutions $U(x,t;C)$ have constant mass

$$M = \int U(x,t;C)\,dx$$

which is determined by the free constant C.

The asymptotic behaviour of any solution u of the Cauchy problem is described in terms of the Barenblatt solution U_M with the same mass as u, as stated in the next two theorems.

Theorem 2.5.2. *Let $u(x,t)$ be the unique solution of the Cauchy problem (2.5.10) with initial data $u_0 \in L^1(\mathbb{R}^n)$ and let U_M be the Barenblatt solution with the same mass as u_0. Then, as $t \to \infty$, we have*

$$\lim_{t\to\infty} \|u(t) - U_M(t)\|_{L^1(\mathbb{R}^n)} = 0.$$

Theorem 2.5.3. *Convergence also holds uniformly in the proper scale*

$$\lim_{t\to\infty} t^{\alpha}\, \|u(t) - U_M(t)\|_{L^\infty(\mathbb{R}^n)} = 0$$

with α given by (2.5.13). Moreover, for every $p \in (1,\infty)$ we have

$$\lim_{t\to\infty} t^{\alpha(p)}\, \|u(t) - U_M(t)\|_{L^p(\mathbb{R}^n)} = 0$$

with $\alpha(p) = \alpha(p-1)/p$.

For a number of different proofs of the above theorems we refer the reader to Vazquez [127]. These results extend to exponents $m > 1$ the well-known convergence of the solutions of the Cauchy problem to the heat equation towards the Gaussian kernel, which implies the asymptotic behaviour

$$u(x,t) \sim \frac{M}{(4\pi t)^{n/2}} \exp(-x^2/4t)$$

holding under the condition $u_0 \geq 0$, $\int_{\mathbb{R}^n} u_0(x)\,dx = M < \infty$.

The first result on the asymptotic behaviour for the Cauchy problem (2.5.10), for $m > 1$, in one space dimension $n = 1$ and compactly supported initial data was shown by S. Kamin [96]. Its extension to dimensions $n \geq 2$ was given by Friedman and Kamin [71]. For a detailed analysis of further results and proofs we refer the reader to [127].

Chapter 3
The Cauchy problem for fast diffusion

This chapter is concerned with the solvability of the Cauchy problem for equations of the form

$$\frac{\partial u}{\partial t} = \Delta\varphi(u) \tag{3.0.1}$$

where the nonlinearity $\varphi(u)$ satisfies certain growth assumptions which correspond to fast diffusion.

In Section 3.1 we shall establish the existence of solutions, existence of initial trace and uniqueness of solutions of the Cauchy problem for equation (3.0.1) in the super-critical fast diffusion case $\varphi \in \mathcal{F}_a$ (as defined in the Introduction) generalizing the pure power case $\varphi(u) = u^m$, $(n-2)/n < m < 1$.

In Section 3.2 we shall study the Cauchy problem for equation (3.0.1) in the special case where $\varphi(u) = \log u$. This is the limiting fast diffusion equation $u_t = \Delta u^m/m$, when $m \to 0$. In critical dimension $n = 2$ this equation represents the evolution of the conformally equivalent metric g with $ds^2 = u\,(dx^2 + dy^2)$ under the Ricci flow and has special interesting features.

In Section 3.3 we shall give a summary of results on the solvability of the Cauchy problem for equation (3.0.1) in the sub-critical case $\varphi(u) = u^m$, $m < (n-2)/n$ as well as in the super-fast diffusion case $\varphi(u) = u^m/m$, $m < 0$.

3.1 The Cauchy problem for super-critical fast diffusion

In this section we shall study the class of non-negative continuous weak solutions of equation

$$\frac{\partial u}{\partial t} = \Delta\varphi(u) \quad \text{on } \mathbb{R}^n \times (0, T) \tag{3.1.1}$$

with $0 < T \le \infty$. The nonlinearity φ is assumed to belong to the class $\mathcal{F}_a$, $a \in (0, 1)$, defined by the conditions:

(i) $\varphi\colon [0, \infty) \to \mathbb{R}$ is a continuous non-negative function with $\varphi(0) = 0$.

(ii) for all $u > 0$, φ satisfies the growth conditions

$$a \le \frac{u\,\varphi'(u)}{\varphi(u)} \le a^{-1}. \tag{3.1.2}$$

(iii.1) There exist $u_0 > 0$ such that for $u \ge u_0$

$$\frac{n-2}{n} + a \le \frac{u\,\varphi'(u)}{\varphi(u)} \le 1 - a. \tag{3.1.3}$$

(iii.2) $u_0 = 1$ and $\varphi(1) = 1$.

We shall present results which were shown in [48] and completely classify the class of continuous weak solutions of (3.1.1). The Cauchy problem for equation (3.1.1) with $\varphi(u) = u^m$, $(n-2)/n < m < 1$ and L^1_{loc} initial data was studied by Herrero and Pierre in [81].

We begin with some preliminary estimates. We observe first that the growth assumptions (3.1.2) and (3.1.3) imply the existence of numbers μ, ν and γ such that $0 < \mu < 1$, $0 < \nu < +\infty$ and $\gamma > (n-2)/n$ which depend only on a, so that, if $\varphi \in \mathcal{F}_a$,

$$\begin{aligned} &\text{(i)} \quad \varphi(u) \leq u^{\mu}, \quad u \geq 1, \\ &\text{(ii)} \quad \varphi(u) \leq u^{\nu}, \quad 0 < u < 1, \\ &\text{(iii)} \quad \varphi(u) \geq u^{\gamma}, \quad u \geq 1. \end{aligned} \tag{3.1.4}$$

We shall also need the following elementary estimate:

$$\int_b^a \frac{du}{A\,u^{\beta} + B\,u^{\mu}} \simeq \begin{cases} \frac{1}{B}\,[a^{1-\mu} - b^{1-\mu}], & Ca > 2, \\ \frac{1}{A}\,[a^{1-\beta} - b^{1-\beta}], & Ca \leq 2, \end{cases} \tag{3.1.5}$$

where $0 < \beta < \mu < 1$, $A > 0$, $B > 0$, $C = (B/A)^{1/(\mu-\beta)}$, and as usual $\simeq$ means that the ratio is bounded above and below. (3.1.5) remains valid if $\beta = \mu$, provided $A = B$. Let now μ, ν be as in (3.1.4), and define $\beta = \min\{\mu, \nu\}$. Then $0 < \beta \leq \mu < 1$, and

$$\varphi(u) \leq u^{\beta}, \quad 0 < u < 1. \tag{3.1.6}$$

Following Herrero and Pierre [81], for any number θ, $0 < \theta < 1$, and $\psi \in \mathcal{S}(\mathbb{R}^n)$, $\psi \geq 0$ (Schwartz class) we set

$$C_{\theta}(\psi) = \left[\int_{\mathbb{R}^n} |\Delta\psi|^{1/(1-\theta)} \psi^{-\theta/(1-\theta)} \right]^{1-\theta}. \tag{3.1.7}$$

Clearly $C_{\theta}(\psi)$ could be infinite, but if $C_{\theta}(\psi) < \infty$ the same is true for $C_{\theta}(\psi_{x_0,R})$, where, for $x_0 \in \mathbb{R}^n$ and $R > 0$

$$\psi_{x_0,R}(x) = R^{-n}\,\psi\left(\frac{x - x_0}{R}\right). \tag{3.1.8}$$

Moreover, once β, μ are fixed as above, we can find $0 \leq \bar{\psi} \leq 1$,

$$\operatorname{supp} \bar{\psi} \subset \{|x| < 2\},$$

$\bar{\psi} \equiv 1$ on $\{|x| < 1\}$, so that $C_{\beta}(\bar{\psi}) < +\infty$, $C_{\mu}(\bar{\psi}) < +\infty$. (See the remarks after (3.6) in [81]). We shall fix such a $\bar{\psi}$ for the remainder of this section.

Lemma 3.1.1. *Let u be a continuous non-negative weak solution of equation* (3.1.1) *with $\varphi \in \mathcal{F}_a$. Let β, μ be as in* (3.1.6) *and* (3.1.4) *respectively, and let $\psi \in C_0^\infty(\mathbb{R}^n)$ be such that $A = C_\beta(\psi) < +\infty$, $B = C_\mu(\psi) < +\infty$. Let*

$$G(b) = \int_0^b \frac{du}{Au^\beta + Bu^\mu}$$

and

$$f(t) = \int_{\mathbb{R}^n} u(x,t)\,\psi(x)\,dx.$$

Then

$$|G(f(t)) - G(f(s))| \le |t-s| \quad \textit{for } 0 < s,t < T. \tag{3.1.9}$$

Proof. By the approximation result Corollary 1.1.2 of Chapter 1, the following formal calculation is justified.

$$f'(t) = \int_{\mathbb{R}^n} \varphi(u(x,t))\,\Delta\psi(x)\,dx$$

and hence

$$\begin{aligned}
|f'(t)| &\le \int_{u\le 1} \varphi(u(x,t))\,|\Delta\psi|\,dx + \int_{u>1} \varphi(u(x,t))\,|\Delta\psi|\,dx \\
&\le \int_{u\le 1} u^\beta(x,t)\,|\Delta\psi|\,dx + \int_{u>1} u^\mu(x,t)\,|\Delta\psi|\,dx \\
&\le C_\beta(\psi)\,f(t)^\beta + C_\mu(\psi)\,f(t)^\mu
\end{aligned}$$

(by Hölder's inequality). The lemma now follows by integrating the differential inequality. □

Corollary 3.1.2. *Let u be as in Lemma* 3.1.1, *$\bar\psi$ as above. Then, there exists $C_1 > 0$, $C_2 > 0$, which depend only on β, μ, $\bar\psi$ and n such that, if $0 < s,t < T$, and*

$$\max\left\{\int_{\mathbb{R}^n} u(x,t)\,\bar\psi_{x_0,R}(x)dx, \int_{\mathbb{R}^n} u(x,s)\,\bar\psi_{x_0,R}(x)\,dx\right\} \le C_1 \tag{3.1.10}$$

then

$$\begin{aligned}
\left|\left[\int_{\mathbb{R}^n} u(x,t)\bar\psi_{x_0,R}(x)\,dx\right]^{1-\beta} - \left[\int_{\mathbb{R}^n} u(x,s)\,\bar\psi_{x_0,R}(x)\,dx\right]^{1-\beta}\right| \\
\le C_2\,|t-s|/R^2
\end{aligned} \tag{3.1.11}$$

while if the max *in* (3.1.10) *is bigger than C_1,*

$$\begin{aligned}
\left|\left[\int_{\mathbb{R}^n} u(x,t)\,\bar\psi_{x_0,R}(x)\,dx\right]^{1-\mu} - \left[\int_{\mathbb{R}^n} u(x,s)\,\bar\psi_{x_0,R}(x)\,dx\right]^{1-\mu}\right| \\
\le C_2\,|t-s|/R^2.
\end{aligned} \tag{3.1.12}$$

Proof. Note that $C_\theta(\bar{\psi}_{x_0,R}) = R^{-2}C_0(\bar{\psi})$. The corollary now follows from (3.1.5) and (3.1.9).

We are now ready to establish the existence of a trace as $t \downarrow 0$.

Theorem 3.1.3. *Let u be a continuous non-negative weak solution of* (3.1.1) *with $\varphi \in \mathcal{F}_a$. Then there exists a unique locally finite Borel measure μ on $\mathbb{R}^n$, such that*

$$\lim_{t\downarrow 0} \int_{\mathbb{R}^n} u(x,t)\,\psi(x)\,dx = \int_{\mathbb{R}^n} \psi(x)\,d\mu(x),$$

for all $\psi \in C_0^\infty(\mathbb{R}^n)$.

Proof. Let $s = T/2$ and consider $0 < t < T/2$. The continuity of u in $\mathbb{R}^n \times (0,T)$ together with Corollary 3.1.2 shows that $\{u(x,t)\}$ has locally uniformly bounded mass. Hence, given $\{t_j\} \to 0$, we can find a subsequence $\{t_{r_j}\} \to 0$, and a locally finite Borel measure μ such that

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} u(x,t_{r_j})\psi(x)dx = \int_{\mathbb{R}^n} \psi(x)d\mu(x)$$

for all $\psi \in C_0^\infty(\mathbb{R}^n)$.

Suppose that $\{s_j\} \to 0$, and let us consider the corresponding $\{s_{\theta_j}\}$ and $\tilde{\mu}$. We want to show that $\mu = \tilde{\mu}$. However, once again, Corollary 3.1.2 applied to $t = t_{r_j}$, $s = s_{\theta_j}$ shows that, for all x_0, R,

$$\int \bar{\psi}_{x_0,R}\,d\mu = \int \bar{\psi}_{x_0,R}\,d\tilde{\mu}.$$

The non-triviality of $\bar{\psi}$ now implies that $d\mu = d\tilde{\mu}$, and the theorem follows. □

We now turn our attention to the existence of solutions in $\mathbb{R}^n \times (0,\infty)$.

Theorem 3.1.4. *Let μ be a locally finite Borel measure on $\mathbb{R}^n$. Then for any $\varphi \in \mathcal{F}_a$, there exists a continuous weak solution u to $\partial u/\partial t = \Delta\varphi(u)$ in $\mathbb{R}^n \times (0,\infty)$, such that*

$$\lim_{t\downarrow 0} \int_{\mathbb{R}^n} u(x,t)\,\eta(x)\,dx = \int_{\mathbb{R}^n} \eta(x)\,d\mu(x) \tag{3.1.13}$$

for all $\eta \in C_0^\infty(\mathbb{R}^n)$.

Proof. Our starting point is the following classical result (Theorem 1.6.1, Chapter 1): for any $u_0 \in C_0^\infty(\mathbb{R}^n)$, there exists $u \in C([0,\infty); L^1(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n \times [0,\infty))$, which is a weak solution of $\partial u/\partial t = \Delta\varphi(u)$ in $\mathbb{R}^n \times (0,\infty)$ and such that $u(x,0) = u_0(x)$. Moreover, u is continuous in $\mathbb{R}^n \times (0,\infty)$.

We next claim that if $R \geq 1$, $\int_{B_{4R}} u_0\,dx \leq M$ then the corresponding u are equicontinuous on compact subsets of $B_R \times (0,\infty)$. To establish this, note that Theorem 1.4.3

in Chapter 1 establishes the local uniform boundedness in $B_R \times [R^2, +\infty)$, and hence the equicontinuity result, Theorem 1.5.1 in Chapter 1 gives the equicontinuity there. If $t < R^2$, choose $r^2 = t < R^2$, and apply the same argument to $B_r \times [r^2, +\infty)$ and its translates, to obtain the desired equicontinuity.

Finally, note that if $\eta \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp} \eta \subset B_R$, $\int_{B_{4R}} u_0 \, dx \le M$, and u is as above, then

$$\left| \int \eta(x)[\, u(x,t) - u_0(x)\,]\, dx \right| \le t\, C_{M,\eta}. \tag{3.1.14}$$

In fact

$$\int \eta(x)[u(x,t) - u_0(x)]\, dx = \int_0^t \int_{\mathbb{R}^n} \Delta\eta(x)\varphi(u(x,s))\, dx ds$$

and so, by (3.1.4) (i), the left-hand side of (3.1.14) is bounded by

$$\int_0^t \int_{\mathbb{R}^n} |\Delta\eta(x)|\, dx\, ds + \int_0^t \int_{\mathbb{R}^n} |\Delta\eta(x)| u^\mu(x,s)\, dx\, ds, \quad 0 < \mu < 1.$$

An application of Hölder's inequality and Corollary 3.1.2 establishes (3.1.14).

Fix now the measure μ, and pick $u_{0,k} \in C_0^\infty(\mathbb{R}^n)$ such that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} u_{0,k}(x)\eta(x)\, dx = \int \eta(x)\, d\mu(x).$$

We can choose $u_{0,k}$ so that, in addition

$$\int_{B_R} u_{0,k}(x)\, dx \le M_R$$

where M_R is independent of k. Let u_R be the corresponding solutions, constructed at the beginning of the proof. By equicontinuity (after possibly passing to a subsequence, which we still denote $\{u_k\}$), there exists a continuous weak solution u of $\partial u/\partial t = \Delta\varphi(u)$ such that the u_k converge to u, uniformly on compact subsets of $\mathbb{R}^n \times (0, \infty)$. The inequality (3.1.14) now establishes (3.1.13). □

Remark. The results in [25] show that, unless the left-hand inequality of the growth condition (3.1.3) is verified (in the pure power case $\varphi(u) = u^m$), Theorem 3.1.4 fails when μ is the delta mass at the origin. As our proof shows, this is because of the lack of an $L^\infty - L^1$-regularizing effect, as in Theorem 1.4.3 in Chapter 1.

Finally, we turn to the uniqueness of the solution constructed in Theorem 3.1.4. The general strategy is the one developed in [45] for the porous medium equation and already presented in the proofs of Lemma 2.2.1 and Theorem 2.2.2. We start out with a version of the maximum principle. Its proof follows closely that of Lemma 2.2.1.

Lemma 3.1.5. *Let u, v be continuous weak solutions of (3.1.1), with $\varphi \in \mathcal{F}_a$. Assume that*

$$\lim_{t\downarrow 0} \int_{|x|<R} [v(x,t) - u(x,t)]^+\, dx = 0 \tag{3.1.15}$$

for all $R > 0$, where $A^+ = \max\{A, 0\}$. Then $v \le u$ in S_T.

Proof. Let $w = v - u$ and let q denote the characteristic function of the set where $u < v$. If u, v and φ are smooth, Kato's inequality [98] shows that

$$\Delta[\varphi(v) - \varphi(u)]^+ \geq q\, \Delta[\varphi(v) - \varphi(u)].$$

Also

$$\frac{\partial w^+}{\partial t} = q\, \frac{\partial w}{\partial t}.$$

By (3.1.3), (3.1.4) and (3.1.6) we see that

$$[\varphi(v) - \varphi(u)]^+ \leq C\,\{\,([v-u]^+)^\mu + ([v-u]^+)^\beta\,\}.$$

Hence, (still under the assumption that u, v, φ are smooth) we have, for $\psi \in C_0^\infty(\mathbb{R}^n)$, $\psi \geq 0$

$$\begin{aligned}
\frac{d}{dt}\int \psi(x) w^+(x,t)dx &\leq \int \psi(x)\Delta[\varphi(v) - \varphi(u)]^+\, dx \\
&\leq \int |\Delta\psi(x)|\, [\varphi(v) - \varphi(u)]^+\, dx \\
&\leq C \int |\Delta\psi(x)|\, \{([v-u]^+)^\mu + ([v-u]^+)^\beta\} \\
&\leq C\, C_\mu(\psi) \left(\int \psi(x)\, w^+(s,t)\, dx\right)^\mu \\
&\qquad + C\, C_\beta(\psi) \left(\int \psi(x)\, w^+(x,t)\, dx\right)^\beta.
\end{aligned}$$

Integrating this (one-sided) differential inequality in a manner similar to the proof of Lemma 3.1.1, using Corollary 1.1.2 of Chapter 1 to justify the formal argument above, and using (3.1.15) we see that, if $C_\mu(\psi) < \infty$ and $C_\beta(\psi) < \infty$, there are constants C_1, C_2 such that, if

$$\frac{1}{R^n}\int_{|x-\xi|<R} [v(x,t) - u(x,t)]^+\, dx \leq C_1$$

then

$$\left(\frac{1}{R^n}\int_{|x-\xi|<R} [v(x,t) - u(x,t)]^+ dx\right)^{1-\beta} \leq C_2\, t\, R^{-2}$$

while if the above quantity exceeds C_1, then

$$\left(\frac{1}{R^n}\int_{|x-\xi|<R} [v(x,t) - u(x,t)]^+\, dx\right)^{1-\mu} \leq C_2\, t\, R^{-2},$$

for any $R > 0$, $\xi \in \mathbb{R}^n$.

Let now

$$h(x,t) = \int_0^t [\varphi(v) - \varphi(u)]^+\, dx.$$

It is easy to see that, for each $t > 0$, h is a subharmonic function of x. Hence, for $\xi \in \mathbb{R}^n$,

$$\begin{aligned} h(\xi,t) &\le \frac{1}{\omega_n R^n}\int_{B_R(\xi)} h(x,t)dx \\ &\le \frac{1}{\omega_n R^n}\int_0^t\int_{B_R(\xi)} [\varphi(v)-\varphi(u)]^+ \\ &\le \frac{C}{R^n}\int_0^t\int_{B_R(\xi)} \{([v-u]^+)^\mu + ([v-u]^+)^\beta\} \\ &\le \frac{C}{R^n}\int_0^t\left[R^{n(1-\mu)}\left(\int_{B_R(\xi)}[v-u]^+\right)^\mu + R^{n(1-\beta)}\left(\int_{B_R(\xi)}[v-u]^+\right)^\beta\right] \\ &\le C\int_0^t s^{\mu/(1-\mu)}R^{-2\mu/(1-\mu)} + s^{\mu/(1-\beta)}R^{-2\mu/(1-\beta)}\,ds \\ &\quad + C\int_0^t s^{\beta/(1-\beta)}R^{-2\beta/(1-\beta)} + s^{\beta/(1-\mu)}R^{-2\beta/(1-\mu)}\,ds. \end{aligned}$$

For fixed t, this tends to 0 as $R \to \infty$, and hence $h(x,t) \equiv 0$, which establishes the lemma. □

Our general uniqueness result now follows from the approximation procedure in [45], Lemma 3.1.5 and the following uniqueness result due to M. Pierre [113].

Theorem 3.1.6. *Let u, v be continuous non-negative weak solutions of $\partial z/\partial t = \Delta\varphi(z)$ on $\mathbb{R}^n \times (0,\infty)$ with $\varphi \in \mathcal{F}_a$. Assume that*

$$\sup_{t>0}\int [u(x,t)+v(x,t)]\,dx \le C$$

for some $C > 0$ and that $u, v \in L^\infty(\mathbb{R}^n \times [\tau,\infty))$ for each $\tau > 0$. If

$$\lim_{t\downarrow 0}\int [u(x,t)-v(x,t)]\,\eta(x)\,dx = 0$$

for any $\eta \in C_0^\infty(\mathbb{R}^n)$, then $u \equiv v$.

The reader may check that the proof of Theorem 2.4.1 in Chapter 2 in the case of slow diffusion $\varphi \in \mathcal{S}_a$ also applies in the fast diffusion case $\varphi \in \mathcal{F}_a$ with the obvious changes.

We are now ready to prove the following uniqueness theorem.

Theorem 3.1.7. *Let u, v be continuous weak solutions of (3.1.1), with $\varphi \in \mathcal{F}_a$. Assume that*

$$\lim_{t\downarrow 0}\int u(x,t)\,\eta(x)\,dx = \lim_{t\downarrow 0}\int v(x,t)\,\eta(x)\,dx$$

for all $\eta \in C_0^\infty(\mathbb{R}^n)$. Then $u \equiv v$ in S_T.

Proof. Let μ be the locally finite Borel measure on $\mathbb{R}^n$ which is the initial trace of u and exists according to Theorem 3.1.3. Pick $h \in C_0^\infty(\mathbb{R}^n)$, $0 \le h \le 1$, and let $w(x,t,h)$ be a solution in $\mathbb{R}^n \times (0,\infty)$, with initial trace $h\,\mu$. By Theorem 3.1.4 at least one such w exists. Moreover, the L^∞ a priori bound, Theorem 1.4.3 in Chapter 1, shows that any such w belongs to $L^\infty(\mathbb{R}^n \times [\tau_1, \tau_2])$, for each $0 < \tau_1 < \tau_2 < +\infty$. We now want to show that, for any such w,

$$\sup_{t>0} \int_{\mathbb{R}^n} w(x,t,h)\,dx < +\infty. \tag{3.1.16}$$

In fact, Corollary 3.1.2 easily implies that for each $t > 0$, $R > 0$,

$$\begin{aligned} &\int_{R/2<|x|<R} w(x,t,h)\,\frac{dx}{|x|^{n-2}} \\ &\quad \le \int_{R/4<|x|<2R} h(x)\,\frac{d\mu(x)}{|x|^{n-2}} + C\,t^{\frac{1}{1-\beta}} R^{2-\frac{2}{1-\beta}} + C\,t^{\frac{1}{1-\mu}} R^{2-\frac{2}{1-\mu}} \end{aligned} \tag{3.1.17}$$

and hence

$$\int_{|x|>1} w(x,t,h)\,\frac{dx}{|x|^{n-2}} < +\infty. \tag{3.1.18}$$

Pick now $\psi(x) \in C^\infty(\mathbb{R}^n)$, $\psi(x) > 0$, $\psi(0) = 1$, ψ bounded, $\Delta\psi \le 0$, $\psi(x) \le C/|x|^n$ for $|x| > 1$, $|\nabla\psi(x)| \le C/|x|^{n-1}$ for $|x| > 1$, and let $\psi_R(x) = \psi(x/R)$. Let $\theta_N(x)$ be a $C_0^\infty(\mathbb{R}^n)$ function, $0 \le \theta_N \le 1$, $\theta_N \equiv 1$ for $|x| < N$, $\operatorname{supp}\theta_N \subset \{|x| < 2N\}$, $|\nabla\theta_N| \le C/N$, $|\Delta\theta_N| \le C/N^2$. Then, for $0 < s < t < +\infty$ (with $w(x,t) = w(x,t,h)$),

$$\begin{aligned} &\int w(x,t)\,\psi_R(x)\,dx - \int w(x,s)\,\psi_R(x)\,dx \\ &\quad = \lim_{N\to\infty} \int [w(x,t) - w(x,s)]\,\theta_N(x)\,\psi_R(x)\,dx \\ &\quad = \lim_{N\to\infty} \int_s^t \int_{\mathbb{R}^n} \varphi(w(x,r))\,\Delta[\theta_N(x)\,\psi_R(x)]\,dx\,dr \\ &\quad \le \overline{\lim_{N\to\infty}} \int_s^t \int_{\mathbb{R}^n} \varphi(w(x,r))\,[\Delta\theta_N(x)\psi_R(x) + 2\nabla\theta_N(x)\nabla\psi_R(x)]\,dx\,dr. \end{aligned}$$

If we now use the pointwise estimates for ψ_R, $\nabla\psi_R$, $\Delta\theta_N$, $\nabla\theta_N$, the support properties of the last two functions, (3.1.4), (3.1.17) and the boundedness of w in $\mathbb{R}^n \times [s,t]$, we see that the above $\overline{\lim}_{N\to\infty}$ is non-positive. Hence

$$\int w(x,t)\psi_R(x)\,dx \le \int w(x,s)\psi_R(x)\,dx.$$

Moreover, (3.1.17) and our pointwise bounds on ψ show that

$$\lim_{s\to 0} \int w(x,s)\,\psi_R(x)\,dx = \int h(x)\psi_R(x)\,d\mu(x).$$

The last expression is bounded, independently on R, and so

$$\int w(x,t)\,\psi_R(x)\,dx \le C.$$

Hence, Fatou's Lemma implies (3.1.16) by letting $R \to \infty$. Pierre's uniqueness result, Theorem 3.1.6, now shows there is only one such $w(x,t,h)$.

Claim. We have

$$w(x,t,h) \le u(x,t). \tag{3.1.19}$$

Indeed, let $U_\varepsilon(x,t)$ be a solution with initial data $h(x)\,u(x,\varepsilon)$. Theorem 3.1.4 and the above argument show that there exists exactly one such U_ε, and that

$$\sup_{t>0}\int_{\mathbb{R}^n} U_\varepsilon(x,t)\,dx \le C \sup_{0<\varepsilon<T/2}\int h(x)\,u(x,\varepsilon)\,dx \le C_h \tag{3.1.20}$$

where the last inequality is a consequence of Corollary 3.1.2. Moreover, for smooth η, we have that

$$\lim_{\varepsilon\to 0}\int \eta(x)\,h(x)\,u(x,\varepsilon)dx = \int \eta(x)\,h(x)\,d\mu(x).$$

Note now that $\lim_{\varepsilon\to 0} U_\varepsilon(x,t) = w(x,t,h)$. In fact, (3.1.20), Theorem 1.4.3 in Chapter 1 and the equicontinuity result, Theorem 1.5.1 in Chapter 1, show that $\{U_\varepsilon(x,t)\}$ is equicontinuous on compact subsets of $\mathbb{R}^n \times (0,\infty)$. Let $\{U_{\varepsilon_j}(x,t)\}$ be a subsequence, which converges uniformly on compact subsets of $\mathbb{R}^n \times (0,\infty)$ to $\hat{w}(x,t)$. Clearly, $\hat{w}(x,t)$ is a continuous weak solution of $\partial u/\partial t = \Delta\varphi(u)$ in $\mathbb{R}^n \times (0,\infty)$. We claim that for any $\eta \in C_0^\infty(\mathbb{R}^n)$

$$\lim_{t\downarrow 0}\int \hat{w}(x,t)\,\eta(x)\,dx = \int h(x)\,\eta(x)\,d\mu(x). \tag{3.1.21}$$

In fact, the argument leading to (3.1.14), together with the second inequality in (3.1.20) show that

$$\left|\int \eta(x)\,[U_\varepsilon(x,t) - h(x)u(x,\varepsilon)]\,dx\right| \le t\,C_h$$

which proves (3.1.21). But the uniqueness of $w(x,t,h)$ then shows that $\hat{w}(x,t) = w(x,t,h)$ and hence $\lim_{\varepsilon\to 0} U_\varepsilon(x,t) = w(x,t,h)$. The function $U_\varepsilon(x,t)$ belongs to $C([0,\infty); L^1(\mathbb{R}^n))$ (see [111]), so that

$$\lim_{t\downarrow 0}\int_{\mathbb{R}^n} |U_\varepsilon(x,t) - h(x)u(x,\varepsilon)|\,dx = 0.$$

By the continuity of $u(x,t)$

$$\lim_{t\downarrow 0}\int_K |u(x,t+\varepsilon) - u(x,\varepsilon)|\,dx = 0,$$

for each $K \subset\subset \mathbb{R}^n$. Hence, by Lemma 3.1.5, $U_\varepsilon(x,t) \le u(x,t+\varepsilon)$, and thus our claim, inequality (3.1.19), follows.

Pick now $0 \le h_j \le h_{j+1} \le 1, h_j \in C_0^\infty(\mathbb{R}^n), \lim_{j\to\infty} h_j(x) \equiv 1$. By the construction of $w(x,t,h)$ we have

$$w(x,t,h_j) \le w(x,t,h_{j+1}).$$

Moreover, by (3.1.19), $w(x,t,h_j) \le u(x,t)$. Let $w_\infty(x,t)$ be a limit of some subsequence of $w(x,t,h_j)$, which exists and is a continuous weak solution, by equicontinuity. The solution $w_\infty(x,t)$ has a trace as $t \downarrow 0$ by Theorem 3.1.3. The trace is between $h_j d\mu$ and $d\mu$ for each j, and hence it equals $d\mu$. Since $w_\infty \le u$,

$$\lim_{t\downarrow 0} \int_{|x|<R} |u(x,t) - w_\infty(x,t)| = 0$$

for all $R > 0$. By Lemma 3.1.5, $u(x,t) = w_\infty(x,t)$. Similarly, $v(x,t) = w_\infty(x,t)$, and hence $u(x,t) = v(x,t)$. □

We finish this section with the following existence and uniqueness result.

Theorem 3.1.8. *Let u be a continuous weak solution of (3.1.7), with $\varphi \in \mathcal{F}_a$. Then there exists a unique $\hat{u}$ in $S_\infty = \mathbb{R}^n \times (0,\infty)$, which is a continuous weak solution of $\partial u/\partial t = \Delta\varphi(u)$ in S_∞, with $u = \hat{u}$ in S_T.*

Proof. Let μ be the trace of u given by Theorem 3.1.3, and let $\hat{u}$ be the corresponding solution in S_∞, constructed in Theorem 3.1.4. By Theorem 3.1.7, $u = \hat{u}$ in S_T. □

3.2 The Cauchy problem for logarithmic fast diffusion

We consider in this section the Cauchy problem for the logarithmic fast diffusion equation

$$\left.\begin{aligned} \partial u/\partial t &= \Delta \log u \quad && \text{in } \mathbb{R}^n \times [0,T), \\ u(x,0) &= u_0 && \text{on } \mathbb{R}^n, \end{aligned}\right\} \tag{3.2.1}$$

with $T > 0$ and initial data u_0 non-negative and locally integrable.

Equation (3.2.1) arises in a number of physical applications. P. L. Lions and G. Toscani [105] have shown that (3.2.1) arises as a singular limit for finite velocity Boltzmann kinetic models and Kurtz [102] has shown that it describes the limiting density distribution of two gases moving against each other and obeying the Boltzmann equation. In dimension $n = 2$ equation (3.2.1) arises also as a model for long Van-der-Waals interactions in thin films of a fluid spreading on a solid surface, if certain nonlinear fourth order effects are neglected [64], [63], [22], [23].

Equation (3.2.1) can be understood as the formal limit, as $m \to 0$, of the fast diffusion equation

$$\frac{\partial u}{\partial t} = \Delta\left(\frac{u^m}{m}\right).$$

Let us note that equation (3.2.1) corresponding to fast diffusion with exponent $m = 0$ in dimension $n = 2$ is critical, since $m = (n - 2)/n$ defines the critical exponent in the sense of [81].

It has been observed by S. Angenent and L. Wu [133], [134] that equation (3.2.1) in the critical dimension $n = 2$ represents the evolution of the conformally equivalent metric g with $ds^2 = u\,(dx^2 + dy^2)$ under the *Ricci flow*, which evolves a metric $ds^2 = g_{ij}\, dx^i dx^j$ by its Ricci curvature R_{ij} with

$$\frac{\partial g_{ij}}{\partial t} = -2\, R_{ij}.$$

The equivalence follows easily from the observation that the conformal metric $g_{ij} = u\, I_{ij}$ has scalar curvature $R = -(\Delta \log u)/u$ and in two dimensions $R_{ij} = \frac{1}{2}\, R\, g_{ij}$. R. Hamilton [77] used the Ricci flow on compact surfaces to obtain new proofs of the uniformization theorem for Riemann surfaces, the topological classification of surfaces, and the topological type of the diffeomorphism group of surfaces. We refer the reader to Hamilton [77], [78], [78], Wu [133], [134] and Cao and Chow [34] for related works.

Our aim in this section is to present recent results regarding the solvability and well-posedness of the Cauchy problem (3.2.1). We shall concentrate our discussion on dimensions $n \geq 2$, giving special emphasis to the critical case $n = 2$, where many interesting phenomena can be observed with important geometric applications.

The one-dimensional problem (3.2.1) is by now well understood, due to a number of works. We refer the reader to the works [69], [115], [98] and their references for detailed results.

Part I. The Cauchy problem (3.2.1) in critical dimension $n = 2$. We shall present a characterization of the solvability of (3.2.1) in the critical dimension $n = 2$ in terms of its initial condition u_0, revealing some of the very interesting and complex structure of its solution set.

The first result shown in [53] (see also [92] and [131] for the radially symmetric case) provides a necessary and sufficient condition for solvability: it states, in particular, that there exists a solution defined up to time $T \leq +\infty$, if and only if $\int_{\mathbb{R}^2} u_0\, dx \geq 4\pi\, T$.

Theorem 3.2.1. *Assume that $u_0 \not\equiv 0$. Then there exists a positive classical solution u to problem (3.2.1) on $\mathbb{R}^2 \times [0, T)$ with*

$$T = \frac{1}{4\pi} \int_{\mathbb{R}^2} u_0(x)\, dx \leq +\infty.$$

In particular, if $\int_{\mathbb{R}^2} u_0\, dx = +\infty$, then (3.2.1) admits a globally defined classical solution.

Conversely, if there is a solution to problem (3.2.1) on $\mathbb{R}^2 \times [0, T)$, then

$$T \leq \frac{1}{4\pi} \int_{\mathbb{R}^2} u_0(x)\, dx. \tag{3.2.2}$$

Moreover,

$$\int_{\mathbb{R}^2} u(x,t)\,dx \le \int_{\mathbb{R}^2} u_0\,dx - 4\pi t \tag{3.2.3}$$

for all $t < T$.

Thus, in particular all solutions to problem (3.2.1) must cease to exist by vanishing before time $(1/4\pi)\int_{\mathbb{R}^2} u_0\,dx$. In addition, the maximal solution of Theorem 3.2.1 is uniquely characterized by its behavior at infinity, as shown by Rodriguez, Vazquez and Esteban in [117], and stated next.

Theorem 3.2.2. *Assume that $u_0 \not\equiv 0$ with $\int_{\mathbb{R}^2} u_0\,dx < +\infty$. Then the maximal solution u of (3.2.1) is uniquely characterized by the minimal decay condition*

$$\frac{1}{u(x,t)} \le O\left(\frac{|x|^2 \log^2 |x|}{2t}\right) \quad \text{as } |x| \to +\infty \tag{3.2.4}$$

locally uniformly in $(0,T)$.

On the other hand, a strong *non-uniqueness* phenomenon takes place: given any integrable initial data u_0, we may find solutions which vanish at any given time less than or equal to $(1/4\pi)\int_{\mathbb{R}^2} u_0$.

Theorem 3.2.3. *Assume $\int_{\mathbb{R}^2} u_0\,dx < +\infty$. Then, for every $\mu > 0$, there exists a solution u_μ to problem (3.2.1) on $\mathbb{R}^2 \times [0,T)$ with*

$$T = T_\mu = \frac{1}{2\pi(2+\mu)} \int_{\mathbb{R}^2} u_0\,dx,$$

with the property that

$$\int_{\mathbb{R}^2} u_\mu(x,t)dx = \int_{\mathbb{R}^2} f(x)\,dx - 2\pi(2+\mu)t, \tag{3.2.5}$$

for all $t < T_\mu$.

For a solution u of (3.2.1) with integrable initial data u_0, we define the function $\phi(t) \in L^1(0,T]$ by the relation

$$\int_{\mathbb{R}^2} u(x,t)\,dx = \int_{\mathbb{R}^2} u_0(x)\,dx - 2\pi \int_0^t \phi(s)\,ds. \tag{3.2.6}$$

The following generalization of Theorem 3.2.3 has been shown in [53] (see also [92]).

Theorem 3.2.4. *Assume that $\int_{\mathbb{R}^2} u_0\,dx < \infty$ and that u solves (3.2.1). Let $\phi(t)$ be the function defined by (3.2.6). Then $\phi \in L^1((0,T])$ and $\phi(t) \ge 2$. Conversely, if $\phi \in L^1((0,T])$ with $\phi(t) \ge 2$, then there exists a solution u to (3.2.1) such that (3.2.6) holds.*

It is shown by Vazquez et al. [131] that in the radially symmetric case the function $\phi(t)$ in (3.2.6) is related to the *flux* of the solution $u = u(r,t)$ at infinity, namely

$$\lim_{r\to+\infty} \frac{r\, u_r(r,t)}{u(r,t)} = -\Phi(t) \tag{3.2.7}$$

with $\Phi(t) = \int_0^t \phi(s)\, ds$.

We shall next present the proofs of Theorems 3.2.1, 3.2.2 and 3.2.3. We refer the reader to [53] for the proof of Theorem 3.2.4. We shall first prove the existence of the intermediate solutions u_μ, corresponding to $\mu > 0$, stated in Theorem 3.2.3. We shall then proceed with the construction and characterization of the maximal solution, stated in Theorems 3.2.1 and 3.2.2.

Because of the flux condition (3.2.7), solutions which satisfy (3.2.5) are expected, at least formally, to decay at infinity as the power $|x|^{-(2+\mu)}$. Having this in mind, we first solve the boundary value problem

$$\left.\begin{aligned} \partial u/\partial t &= \Delta \log u && \text{in } B_R \times [0,\infty), \\ u(x,t) &= R^{-(2+\mu)} && \text{on } \partial B_R \times (0,\infty), \\ u(x,0) &= u_0(x) && x \in B_R. \end{aligned}\right\} \tag{3.2.8}$$

on a sequence of expanding cylinders $B_{R_n} \times [0,\infty)$. We then use specific solutions as barriers in an average sense to show that the limit of those solutions along a subsequence of $R_{n_k} \to \infty$ is a solution which satisfies (3.2.5).

In the next proposition we state the existence of certain special, traveling wave-like solutions to equation $u_t = \Delta \log u$, which will play a main role in the proof of Theorem 3.2.3. These solutions were constructed by Hamilton in [77] following a detailed ODE analysis.

Proposition 3.2.5. *Given any numbers $K > 0$ and $\mu > 0$, there exists a radially symmetric smooth positive solution $v(x,t)$ to equation $u_t = \Delta \log u$ on $\mathbb{R}^2 \times (0,T)$, with $T = K/2\pi(2+\mu)$ such that*

$$\int_{\mathbb{R}^2} v(x,t)\, dx = K - 2\pi(2+\mu)t \quad \textit{for all } t \in [0,T). \tag{3.2.9}$$

Proof. It suffices to show that for each $\mu > 0$, there is a radially symmetric solution $w(r,t)$ to the equation $u_t = \Delta \log u$, with $T = 1$, which satisfies

$$\int_0^\infty w(r,t)\, r\, dr = (1-t)\,(2+\mu)$$

since, for such solution w, the function

$$v(r,t) = \frac{K}{2\pi(2+\mu)}\, w\left(r, \frac{2\pi(2+\mu)}{K}\, t\right)$$

is a solution to (3.2.1) which satisfies (3.2.9). When $\mu = 2$, we just set w to be the explicit solution of the equation

$$w(r,t) = \frac{8(1-t)}{(1+r^2)^2}. \tag{3.2.10}$$

Assume next that $\mu > 2$. We look for a solution w of the form

$$w(r,t) = (1-t)\,\frac{g(\log r - \gamma \log(1-t))}{r^2} \tag{3.2.11}$$

where $\gamma > 0$ is to be determined in terms of μ. A direct computation shows that g must satisfy the differential equation

$$\left\{\frac{g'}{g}\right\}' + g - \gamma g' = 0, \qquad y \in (-\infty, \infty). \tag{3.2.12}$$

This equation has been analyzed in [77]. In particular, it is shown there that given any numbers $\alpha > \beta > 0$, there is a choice of $\gamma > 0$ such that there exists a solution $g(y)$ to (3.2.12) which admits expansions of the form $g(y) = a_1 e^{-\alpha y} + a_2 e^{-2\alpha y} + \cdots$ as $y \to +\infty$ and $g(y) = b_1 e^{\beta y} + b_2 e^{2\beta y} + \cdots$ as $y \to -\infty$.

Let us choose $\beta = 2$ and $\alpha = \mu$. Note that this choice makes the associated w in (3.2.11) non-singular and a solution of $u_t = \Delta \log u$, up to $r = 0$. Moreover, we have that $(g'/g)(+\infty) = -\mu$ and $(g'/g)(-\infty) = +2$. Hence, integrating (3.2.12) we obtain

$$2 + \mu = \int_{-\infty}^{\infty} g(y)\,dy$$

from where relation (3.2.9) readily follows.

Finally, assume $0 < \mu < 2$. In this case we take $\alpha = 2$ and $\beta = -\mu$ and choose $\gamma > 0$ and g as above. Let us now set

$$w(r,t) = (1-t)\,\frac{g(-\log r - \gamma \log(1-t))}{r^2}.$$

This w defines a solution of (3.2.1) with the desired properties, thus finishing the proof. □

Before we proceed with the proof of Theorem 3.2.3, we introduce the operator

$$G_R^*(h)(x) = \int_{\mathbb{R}^2} [G_R(x,y) - G_R(0,y)]\,h(y)\,dy, \tag{3.2.13}$$

where for $R > 0$, $G_R(x,y)$ denotes the Green's function for the ball $B_R = \{x : |x| \leq R\}$, given by

$$G_R(h)(x,y) = \begin{cases} \frac{1}{2\pi}\{\log(|x-y|) - \log(|\frac{|x|\,y}{R} - \frac{R\,x}{|x|}|)\} & x \neq 0, \\ \frac{1}{2\pi}\{\log(|y|) - \log(R)\} & x = 0. \end{cases}$$

Notice that if h is bounded, then

$$\Delta G_R^*(h)(x) = h(x) \quad \text{for all } x \in B_R \tag{3.2.14}$$

while if h is smooth and constant on ∂B_R, then

$$G_R^*(\Delta h)(x) = h(x) - h(0) \quad \text{for all } x \in B_R. \tag{3.2.15}$$

Proof of Theorem 3.2.3. We shall first show the result under the assumption that the initial data u_0 is in $C_0^\infty(\mathbb{R}^2)$. The general case will follow by approximation.

To simplify the notation we shall omit the subscript μ from u and T. For $R > 0$ and $\varepsilon \in (0,1)$, let u_ε^R denote the unique solution to problem (3.2.8) with initial data $u_\varepsilon^R(x,0) = u_0(x) + \varepsilon$. Existence and uniqueness of u_ε^R follow from the standard theory of non-degenerate quasilinear parabolic equations [103].

Consider $G_R^*(u_0)$, as defined in (3.2.13). It is easy to show that if the support of u_0 is contained in the ball B_{R_0}, then for all $R \geq R_0$ and all $x \in B_R$, we have

$$G_R^*(u_0)(x) = \left(\int_{\mathbb{R}^2} u_0 \, dx \right) \cdot \log(|x|+1), \tag{3.2.16}$$

where the function $\theta(x)$ is locally bounded with

$$\theta(x) = o(\log(|x|+1)) \quad \text{as } |x| \to \infty. \tag{3.2.17}$$

For a small number $\delta > 0$, let $v^{T-\delta}$, $v^{T+\delta}$ denote the specific radially symmetric solutions of Proposition 3.2.5, with initial mass

$$\int_{\mathbb{R}^2} v^{T-\delta}(x,0)\, dx = 2\pi(2+\mu)(T-\delta) \quad \text{and} \quad \int_{\mathbb{R}^2} v^{T+\delta}(x,0)\, dx = 2\pi(2+\mu)(T+\delta).$$

Since $\int_{\mathbb{R}^2} u_0 \, dx = 2\pi(2+\mu)T$, it follows from (3.2.16) and (3.2.17) that

$$-l_\delta + G_R^*(v^{T-\delta}(\cdot,0))(x) \leq G_R^*(u_0)(x) \leq G_R^*(v^{T+\delta}(\cdot,0))(x) + l_\delta$$

for some non-negative constant l_δ.

We would like to show that if $u^R = \lim_{\varepsilon\to 0} u_\varepsilon^R$, then there exists a sequence $R_j \uparrow \infty$ and a non-negative constant L_δ, which can be chosen to be independent of R_j, such that

$$\begin{aligned} -L_\delta + G_{R_j}^*(v^{T-\delta}(\cdot,t))(x) &\leq G_{R_j}^*(u^{R_j}(\cdot,t))(x) \\ &\leq G_{R_j}^*(v^{T+\delta}(\cdot,0))(x) + L_\delta. \end{aligned} \tag{3.2.18}$$

for all $|x| \leq R_j$ and $0 \leq t \leq T - 2\delta$. We begin by showing the left hand side of (3.2.18). Set

$$W(x,t) = G_R^*(u_\varepsilon^R(\cdot,t) - v^{T-\delta}(\cdot,t))(x).$$

Using the maximum principle we shall prove that $W(x,t) \geq -L_\delta$, for $|x| \leq R$, $0 \leq t \leq T - 2\delta$. Indeed, since both functions u_ε^R and $v^{T-\delta}$ are constant on ∂B_R, we can compute using (3.2.15) that

$$\frac{\partial W}{\partial t} = G_R^*\big(\Delta[\log u_\varepsilon^R - \log v^{T-\delta}]\big) = a(x,t)\Delta W - b(t) \tag{3.2.19}$$

where

$$a(x,t) = \frac{\log u_\varepsilon^R - \log v^{T-\delta}}{u_\varepsilon^R - v^{T-\delta}}$$

and

$$b(t) = \log u_\varepsilon^R(0,t) - \log v^{T-\delta}(0,t) \leq B$$

with B depending only on $\|u_0\|_\infty$ and δ. Therefore, if we set $W = W + Bt$, then W satisfies the differential inequality

$$\frac{\partial W}{\partial t} \geq a(x,t)\,\Delta W$$

on $B_R \times (0, T - \delta)$. At $t = 0$ we have

$$W(x,0) = W(x,0) \geq -l_\delta.$$

To see what happens on the lateral boundary, we first observe that for the special solution $v^{T-\delta}$ there exist constants $c_1(\delta)$ and $c_2(\delta)$, such that for $0 \leq t \leq T - 2\delta$,

$$\frac{c_1(\delta)}{R^{2+\mu}} \leq v^{T-\delta}(R,t) \leq \frac{c_2(\delta)}{R^{2+\mu}}$$

if $R \geq R_0$, with R_0 sufficiently large. It follows that for $|x| = R \geq R_0$, $0 \leq t \leq T - 2\delta$

$$\frac{\partial W(x,t)}{\partial t} = \log u_\varepsilon^R(x,t) - \log v^{T-\delta}(x,t) - b(t) + B \geq 0$$

if B is chosen sufficiently large, which implies that for $|x| = R$, $0 \leq t \leq T - 2\delta$ we have

$$W(x,t) \geq W(x,0) \geq -l_\delta.$$

We can apply now the maximum principle to conclude that $W(x,t) \geq -l_\delta$. Hence, by letting $\varepsilon \to 0$, we obtain for $0 \leq t \leq T - 2\delta$ the inequality

$$G_R^*(u^R(\cdot,t))(x) \geq G_R^*(v^{T-\delta}(\cdot,t))(x) - L_\delta \tag{3.2.20}$$

with $L_\delta = l_\delta + BT$.

Before we show the right hand side of (3.2.18) we shall first construct the solution u. Taking spherical averages on both sides of (3.2.20) we obtain

$$\int_0^r \frac{ds}{2\pi s} \int_{B_s} u^R(x,t)\,dx \geq \int_0^r \frac{ds}{2\pi s} \int_{B_s} v^{T-\delta}(x,t)\,dx - L_\delta$$

for $0 < r < R$ and $0 < t \le T - 2\delta$. Remembering that the special solutions $v^{T-\delta}$ have the form

$$v^{T-\delta}(x,t) = (T - t - \delta)_+ \frac{g(\log|x| + \log\alpha(t))}{|x|^2},$$

with $\alpha(t) = ((T-\delta)/(T-t-\delta))^\gamma$ and $\int g(y)\,dy = 2+\mu$, and computing that

$$\int_0^r \frac{ds}{2\pi s}\int_{B_s} v^{T-\delta}(x,t)\,dx = (T-t-\delta)_+ \int_0^{r\alpha(t)} \frac{ds}{2\pi s}\int_{B_s} g(\log|x|)\,dx,$$

we deduce that by choosing L_δ sufficiently large we can make

$$\int_0^r \frac{ds}{2\pi s}\int_{B_s} v^{T-\delta}(x,t)\,dx \ge (2+\mu)(T-t-2\delta)\,\log(1+r) - L_\delta,$$

for all $r > 0$. Therefore, we have

$$\int_0^r \frac{ds}{2\pi s}\int_{B_s} u^R(x,t)\,dx \ge (2+\mu)(T-t-2\delta)\,\log(1+r) - L_\delta. \tag{3.2.21}$$

Basic Claim. Given a number $\delta > 0$ and an increasing sequence $R_j \uparrow \infty$, there exists a subsequence, still denoted by $\{R_j\}$, such that the sequence of solutions u^{R_j} converges uniformly on compact subsets of $\mathbb{R}^2 \times (0, T-3\delta]$ to a solution u^δ of (3.2.1).

To show the claim, we begin by observing that there exists a point $x_0 \in \mathbb{R}^2$ such that

$$\limsup_{R_j\to\infty} u^{R_j}(x_0, T-2\delta) > 0. \tag{3.2.22}$$

Indeed, if $\lim_{j\to\infty} u^{R_j}(x, T-2\delta) = 0$, for all $x \in \mathbb{R}^2$, then from Dominated convergence we conclude that

$$\limsup_{j\to\infty}\int_0^r \frac{ds}{2\pi s}\int_{B_s} u^{R_j}(x, T-2\delta)\,dx = 0,$$

for all $r > 0$, which contradicts (3.2.21). It follows from (3.2.22) that we can choose a subsequence, still denoted by $\{R_j\}$, such that

$$u^{R_j}(x_0, T-2\delta) \ge c > 0 \quad \text{for all } j \ge j_0. \tag{3.2.23}$$

For $r > 0$ and $s \in (0, T-3\delta)$ set $K = B_r(x_0) \times (s, T-3\delta]$ and let $j \ge j_0$ be sufficiently large so that the cylinder $B_{R_j} \times (0,T)$ strictly contains K. We shall show that (3.2.23) implies the estimate from below

$$u^{R_j}(y,\tau) \ge c(K) > 0 \quad \text{for all } j \ge j_0,\ (y,\tau) \in K. \tag{3.2.24}$$

This will be a consequence of the following Harnack type estimates, satisfied by all solutions u^{R_j}, namely the inequalities

$$(\pi\rho)^{-2}\int_{B_\rho(y_0)} (-\log u)(x,t)\,dx \le (-\log u)(y_0, T-2\delta) - \log t + C_\delta M\rho^2 \tag{3.2.25}$$

and

$$(-\log u)(y_0, t) \leq (\pi\rho)^{-2} \int_{B_\rho(y_0)} (-\log u)(x, t)\, dx - \frac{M\rho^2}{t}, \tag{3.2.26}$$

holding for all $\rho > 0$ such that $B_\rho(y_0) \subset\subset B_{R_j}(0)$ and $t \in (0, T - 3\delta]$. Here M is an upper bound for the initial data u_0. The estimate (3.2.25) is proven in [124] (Lemma 6) by testing the equation by a suitable Green's function and integrating by parts. The estimate (3.2.26) can be easily proven in a similar manner, by multiplying the inequality $\Delta \log u \leq u/t$ by the Green's function used in [124] and integrating. Since their proof is quite standard we omit the details, referring the reader to [124] for a detailed proof.

From (3.2.23) and (3.2.25) and we obtain

$$\int_{B_\rho(x_0)} (-\log u^{R_j})(x, t)\, dx \leq C(\rho, \delta) - \log t, \quad 0 < t \leq T - 3\delta. \tag{3.2.27}$$

Thus if $(y, \tau) \in K$ and $\rho = 2r$, (3.2.25) and (3.2.26) imply that

$$(-\log u^{R_j})(y, \tau) \leq C(K) \int_{B_\rho(x_0)} (-\log u^{R_j})(x, \tau)\, dx \leq C(K)$$

which yields (3.2.24).

Since the sequence of solutions $\{u^{R_j}\}$ is uniformly bounded from above by $\|u_0\|_\infty$, it follows from (3.2.23) and the classical theory of parabolic equations that the sequence $\{u^{R_j}\}$ is equicontinuous on compact subsets of $\mathbb{R}^2 \times (0, T - 3\delta]$. Hence, there exists a subsequence, still denoted by $\{u^{R_j}\}$, which converges uniformly on compact subsets of $\mathbb{R}^2 \times (0, T - 3\delta]$ to a function u^δ.

It remains to show that u^δ is a solution of (3.2.1) on $\mathbb{R}^2 \times (0, T - 3\delta]$. It is clear that u^δ satisfies the equation (3.2.1) in the distributional sense, since each u^{R_j} does. Also, because $u^\delta \leq \|u_0\|_\infty$, it follows from (3.2.27) that $\log u^\delta \in L^1_{\mathrm{loc}}(\mathbb{R}^2 \times [0, T - 3\delta])$, with

$$\int_{B_r(0)} |\log u^\delta(x, t)|\, dx \leq C(\|u_0\|_\infty, r, \delta)|\log t|.$$

It remains to show that $u^\delta(\cdot, \tau) \to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ as $\tau \to 0$. Indeed, observe first that for $r > 0$, (3.2.27) implies that

$$\left| \int_{B_r(0)} u^\delta(x, \tau)\, dx - \int_{B_r(0)} u_0(x)\, dx \right| \leq C(\delta, r) \int_0^\tau |\log t|\, dt. \tag{3.2.28}$$

Moreover, if for $\varepsilon > 0$, u_ε denotes the unique solution to (3.2.1) with initial data $u_\varepsilon(\cdot, 0) = u_0 + \varepsilon$, then $u^\delta \leq u_\varepsilon$ (see [52], Theorem 1.2) and therefore we have

$$\begin{aligned} \int_{B_r(0)} [u^\delta(x, \tau) - u_0(x)]_+\, dx &\leq \int_{B_r(0)} [u_\varepsilon(x, t) - u_0(x)]_+ \\ &\leq \int_{B_r(0)} |u_\varepsilon(x, \tau) - u_\varepsilon(x, 0)|\, dx + |B_r(0)|\,\varepsilon. \end{aligned} \tag{3.2.29}$$

We then easily conclude that

$$\int_{B_r(0)} |u^\delta(x,\tau) - u_0(x)|\, dx \to 0 \quad \text{as } \tau \to 0$$

which proves the desired result.

In order to construct a solution which is defined up to time T and satisfies (3.2.5) we shall combine (3.2.21) with the following estimate from above

$$\int_0^r \frac{ds}{2\pi s} \int_{B_s} u^\delta(x,t)\, dx \le (2+\mu)(T-t+2\delta)\, \log(r+1) + L_\delta \tag{3.2.30}$$

holding for all $0 < r < R$ and $0 < t \le T - 2\delta$. It is easy to observe that (3.2.30) follows from the right hand side of (3.2.18) by taking spherical averages, computing that

$$\int_0^r \frac{ds}{2\pi s} \int_{B_s} v^{T+\delta}(x,t)\, dx \le (2+\mu)(T-t+2\delta)\, \log(r+1) + L_\delta$$

for all $r > 0$ and letting $R_j \to \infty$. Hence, we need to show the right hand side of (3.2.18). We shall use again the maximum principle. As before, we shall apply the maximum principle to the function

$$Z(x,t) = G_R^*(u_\varepsilon^R(\cdot,t) - v^{T+\delta}(\cdot,t))(x),$$

which satisfies the equation $\partial Z/\partial t = d(x,t)\Delta Z - e(t)$ with

$$d(x,t) = \frac{\log(u_\varepsilon^R(\cdot,t)) - \log(v^{T+\delta}(\cdot,t))}{u_\varepsilon^R(\cdot,t) - v^{T+\delta}(\cdot,t)}$$

and

$$e(t) = \log u_\varepsilon^{R_j}(0,t) - \log v^{T+\delta}(0,t).$$

To bound the coefficient $e(t)$, we notice that from (3.2.24), we have $u_\varepsilon^{R_j}(0, T-2\delta) \ge c$, for some positive constant c depending only on δ. Moreover, each of the solutions $u_\varepsilon^{R_j}$ satisfies the Aronson–Bénilan inequality $u_t \le u/t$, which by integration gives

$$\log u_\varepsilon^{R_j}(0,t) \ge \log u_\varepsilon^{R_j}(0, T-2\delta) - \log(T-2\delta) + \log t \ge C(\delta) + \log t.$$

Hence, $e(t) \ge -E + \log t$, for some constant $E = E(\delta)$ and thus is we set $E(t) = Et + \int_0^t \log s\, ds$ and $Z = Z + E(t)$, then Z satisfies

$$\frac{\partial Z}{\partial t} \le d(x,t)\Delta Z.$$

At $t = 0$ we have $Z(x,0) = Z(x,0) \le l_\delta$. Moreover, by choosing the constant E sufficiently large we can show as we did for W that $\partial Z/\partial t \le 0$ on $|x| = R$, and conclude that $Z \le l_\delta$ on the lateral boundary of $B_R(0) \times (0, T-2\delta)$. Therefore, by the maximum principle $Z \leq l_\delta$ which implies the right hand side of (3.2.18) for $R = R_j$, if L_δ is chosen sufficiently large.

We have shown that the solution u^δ satisfies

$$\begin{aligned} -\frac{L_\delta}{\log(r+1)} &+ (2+\mu)(T-t-2\delta) \\ &\leq \frac{1}{\log(r+1)} \int_0^r \frac{ds}{2\pi s} \int_{B_s} u^\delta(x,t)\,dx \\ &\leq \frac{L_\delta}{\log(r+1)} + (2+\mu)(T-t+\delta) \end{aligned} \tag{3.2.31}$$

for all $r > 0$ and $0 \leq t \leq T - 2\delta$. A simple computation shows that if h is a bounded integrable function then

$$\lim_{r\to\infty} \frac{1}{\log(r+1)} \int_0^r \frac{ds}{2\pi s} \int_{B_s} h(x)\,dx \to \frac{1}{2\pi} \int_{\mathbb{R}^2} h(x)\,dx.$$

Thus setting $u_k = u^\delta\, \chi_{|x|\leq k}$ and using the right hand side of (3.2.31) we can deduce that

$$\int_{|x|\leq k} u_k(x,t)\,dx \leq 2\pi(2+\mu)(T-t-2\delta),$$

for $0 \leq t \leq T - 2\delta$. It follows from the Monotone Convergence Theorem that $\int_{\mathbb{R}^2} u^\delta(x,t)\,dx < \infty$ and therefore we can use again (3.2.31) to conclude that

$$(2+\mu)(T-t-\delta) \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} u^\delta(x,t)\,dx \leq (2+\mu)(T-t+2\delta). \tag{3.2.32}$$

It remains to construct a solution u of the problem (3.2.1) which is defined up to time T and satisfies (3.2.5). To this end, choose a decreasing sequence $\delta_k \downarrow 0$ and starting with any increasing sequence $R_j \uparrow \infty$, let R_j^1 be a subsequence such that $u^{R_j^1} \to u^{\delta_1}$ uniformly on compacts of $\mathbb{R}^2 \times (0, T - 3\delta_1]$. We construct u^{δ_k} inductively. Suppose $u^{\delta_{k-1}} = \lim_{R_j^{k-1}\to\infty} u^{R_j^{k-1}}$, then let R_j^k be a subsequence of R_j^{k-1} such that $u^{R_j^k}$ converges uniformly on compact subsets of $\mathbb{R}^2 \times (0, T - 3\delta_k]$ and set $u^{\delta_k} = \lim_{R_j^k\to\infty} u^{R_j^k}$. By construction we have $u^{\delta_k} = u^{\delta_{k-1}}$ on $\mathbb{R}^2 \times (0, T - 3\delta_{k-1})$ and therefore if we define $u(x,t) = u^{\delta_k}(x,t)$, if $0 < t \leq T - 3\delta_k$, it is clear that u is the desired solution defined on $\mathbb{R}^2 \times (0, T)$. Moreover, it follows from (3.2.32) that u satisfies the important identity (3.2.5).

To remove the assumption that $u_0 \in C_0^\infty$, choose an increasing sequence $u_0^k \in C_0^\infty$, such that $\|u_0^k - u_0\|_{L^1(\mathbb{R}^2)} \to 0$ and let u^k be the solution of the problem (3.2.1) defined on $\mathbb{R}^2 \times (0, T_k)$, with $T_k = 1/(2\pi(2+\mu)) \int_{\mathbb{R}^2} u_0^k$, as constructed above. Each of the u^k satisfies

$$\int_{\mathbb{R}^2} u^k(x,t)\,dx = \int_{\mathbb{R}^2} u_0^k(x)\,dx - 2\pi(2+\mu)t, \quad 0 \leq t \leq T_k. \tag{3.2.33}$$

Moreover, from the previous construction it is easy to deduce that the sequence u^k is increasing. Therefore if we set

$$u(x,t) = \lim_{k\to\infty} u^k(x,t) \quad \text{for } (x,t) \in \mathbb{R}^2 \times (0,T),$$

we can easily conclude from (3.2.33) and monotone convergence that $u(\cdot,t) \in L^1(\mathbb{R}^2)$, for all $t \in (0,T)$ and satisfies (3.2.5). Moreover, since $(\log u)_+ \le u$ and $u \ge u^k$, we have that $\log u \in L^1_{\text{loc}}(\mathbb{R}^2 \times [0,T))$. It follows in a similar manner as above, that u satisfies (3.2.1) in the sense of distributions and that $u(\cdot,\tau) \to u_0$ in $L^1_{\text{loc}}(\mathbb{R}^2)$ as $\tau \to 0$. The proof of the theorem is now complete. □

Proof of Theorem 3.2.1. For simplicity we shall assume that u_0 is uniformly bounded. The general case follows again by an approximation argument. We refer the reader to [52] for the details.

For $\varepsilon > 0$, we let u_ε denote the solution of problem (3.2.1) with initial data $u_0 + \varepsilon$. Standard arguments show that this solution is unique and globally defined in time. Moreover, one may observe that $u_\varepsilon \ge u$ for all $\varepsilon > 0$ (see [52], Theorem 1.2 for a detailed proof of this claim). We shall estimate u_ε. For a radial integrable function g, denote

$$N(g)(r) = \frac{1}{2\pi}\int_0^r \frac{1}{s}\int_{|x|\le s} g(x)\,dx$$

and observe that if $\bar{g}(r)$ denotes the spherical averages of g, then

$$N(\Delta g)(r) = \bar{g}(r) - g(0).$$

Thus, applying the operator N to the equation and then integrating in time on $[0, T-\delta]$, we obtain

$$\begin{aligned} -\int_0^{T-\delta} \overline{\log u_\varepsilon}(r,t)\,dt + \int_0^{T-\delta} \log u_\varepsilon(0,t)dt \\ = -N(u_\varepsilon(\cdot, T-\delta))(r) + N(u_0+\varepsilon)(r). \end{aligned} \tag{3.2.34}$$

Now, since u_ε satisfies the Aronson–Bénilan inequality, $u_t/u \le 1/t$, we have that

$$\begin{aligned} -\int_0^{T-\delta} \log u_\varepsilon(0,t)\,dt \\ \le -(T-\delta)\log u_\varepsilon(0,T-\delta) - \int_0^{T-\delta} \log\left(\frac{s}{(T-\delta)}\right) ds \end{aligned} \tag{3.2.35}$$

Using (3.2.34), (3.2.35) and Jensen's inequality we get

$$\begin{aligned} -\int_0^{T-\delta} \log \bar{u}_\varepsilon(r,s)\,ds \le N(u_0)(r) + C^* + A\varepsilon r^2 \\ \le \frac{\log r}{2\pi}\int_{\mathbb{R}^2} u_0\,dx + o(\log r) + A\varepsilon r^2 \end{aligned} \tag{3.2.36}$$

where the constant C^* is given by the right hand side of (3.2.35). Since $u_\varepsilon(0, T-\delta) > u(0, T-\delta) > 0$, we can choose the constant C^* to be independent of ε. Thus, from (3.2.36) and the Aronson–Bénilan inequality we deduce that

$$\bar{u}_\varepsilon(r, \delta) \geq B\, r^{1/(2\pi(T-2\delta))\int_{\mathbb{R}^2} u_0} \exp(-A\varepsilon r^2 - C) \tag{3.2.37}$$

with constants B and C independent of δ and ε. On the other hand applying again the operator N to the equation and integrating in time on $[0, \delta]$, we obtain

$$\int_0^\delta \overline{\log u_\varepsilon}(r, t) dt \geq \int_0^\delta \log u_\varepsilon(0, t)\, dt + N(\bar{u}_\varepsilon(\,\cdot\,, \delta))(r) - N(u_0 + \varepsilon)(r)$$

and therefore with the use of (3.2.37) we deduce the inequality

$$\int_0^\delta \overline{\log u_\varepsilon}(r, t)\, dt \geq B\, N(r^{1/(2\pi(T-2\delta))\int u_0\, dx} \exp(-A\varepsilon r^2 - C)) - C^* - A\varepsilon r^2$$

where C and C^* can be taken independent of ε. Since the initial data u_0 is uniformly bounded, the left hand side of the above inequality is uniformly bounded independently of ε. Therefore we must have

$$\frac{1}{2\pi(T-2\delta)} \int_{\mathbb{R}^2} u_0\, dx \geq 2.$$

Since δ is arbitrary the result follows in case that u_0 is bounded. In the general case, one proceeds using approximations for u_0 which are bounded as in [52].

We shall show next (3.2.3). Assume that there is a $\bar{t} \in (0, T)$ such that the opposite inequality holds, i.e.,

$$\int_{\mathbb{R}^2} u(x, \bar{t})\, dx > \int_{\mathbb{R}^2} u_0\, dx - 4\pi \bar{t}. \tag{3.2.38}$$

From Theorem 3.2.3 we know that for every $\mu > 0$ there exists a solution v to (3.2.1) defined on $[\bar{t}, \bar{t} + T_\mu]$, whose initial data is $u(\,\cdot\,, \bar{t}\,)$, and where

$$T_\mu = \frac{1}{2\pi(2+\mu)} \int_{\mathbb{R}^2} u(x, \bar{t}\,)\, dx.$$

Thus, the function w defined on $[0, \bar{t} + T_\mu]$ by matching together u and v is a solution to (3.2.1) in this interval, with initial data u_0. But because of (3.2.37), by choosing μ sufficiently small we can make

$$\bar{t} + T_\mu > \frac{1}{4\pi} \int_{\mathbb{R}^2} u_0(x)\, dx,$$

which contradicts (3.2.2). Therefore (3.2.3) must hold true.

We shall now construct a solution with the given initial data, which exists up to time $T = 1/4\pi \int_{\mathbb{R}^2} u_0(x)\, dx$. This is going to be the maximal solution with the given initial data, as actually (3.2.2) indicates. Its construction is fairly straightforward. As in the proof of Theorem 3.2.3 we make first the extra assumption that $u_0 \in C_0^\infty(\mathbb{R}^2)$. For $\varepsilon > 0$, let u^ε denote the solution of the initial value problem (3.2.1) with initial data $u^\varepsilon(x,0) = u_0 + \varepsilon$, defined on $\mathbb{R}^2 \times (0,\infty)$. Then, if for $\mu > 0$, u_μ denotes the solution constructed above and satisfying (3.2.5), we have

$$u_\mu \le u^\varepsilon \quad \text{on } \mathbb{R}^2 \times (0, T_\mu).$$

It follows by the maximum principle that the sequence $\{u^\varepsilon\}$ is monotonic and hence the limit $u = \lim_{\varepsilon\to 0} u^\varepsilon$ exists and for all $\mu > 0$ satisfies

$$u_\mu \le u \quad \text{on } \mathbb{R}^2 \times (0, T_\mu) \tag{3.2.39}$$

Since $T_\mu \uparrow T$ as $\mu \to 0$ it is easy to conclude from (3.2.39) that u is a solution of (3.2.1) on $\mathbb{R}^2 \times (0,T)$, with $T = 1/4\pi \int_{\mathbb{R}^2} u_0(x)\,dx$. Moreover since (3.2.5) holds for all u_μ, $\mu > 0$ implying that

$$\int_{\mathbb{R}^2} u(x,t)\,dx \ge \int_{\mathbb{R}^2} u_0(x)\,dx - 4\pi t, \quad 0 \le t < T. \tag{3.2.40}$$

However, (3.2.3) shows that we must have equality in (3.2.40). Therefore u is the desired solution. We can now use approximation as in the proof of Theorem 3.2.3 to prove the result for any initial data $u_0 \in L^1(\mathbb{R}^2)$.

To finish the proof of the theorem, in the case that $\int_{\mathbb{R}^2} u_0(x)\,dx = \infty$, choose a sequence $u_0^k \uparrow f$, such that $\int_{\mathbb{R}^2} u_0^k < \infty$ and let u^k be the maximal solution to the problem (3.2.1) with initial data u_0^k as constructed above. The solution u^k exists up to time $T_k = 1/4\pi \int_{\mathbb{R}^2} u_0^k$. Since, the sequence u_k is increasing, the limit $u = \lim_{k\to\infty} u^k$ exists, and it is not hard to check that it is indeed a solution of (3.2.1) with initial data u_0, which exists up to time $T = \infty$. The proof of the Theorem 3.2.1 is now complete. □

We now proceed to the proof of Theorem 3.2.2. Its uniqueness assertion is based on the next comparison principle proven in [117].

Proposition 3.2.6. *Let u and v satisfy (3.2.1) on $\mathbb{R}^2 \times (0,T)$ and assume that u satisfies the decay condition (3.2.4) uniformly in t in compact subintervals of $(0,T)$. Then, for any $0 < \tau_1 < \tau_2 < T$ and $0 < m < 1$, there exist positive constants $K = K(\tau_1,\tau_2,m)$ and $R_0 = R_0(\tau_1,\tau_2)$ such that for all $0 < \tau_1 < s < t < \tau_2$ and $R \ge R_0$, we have*

$$\begin{aligned}\left(\int_{|x|<R} [v(x,t) - u(x,t)]_+\,dx\right)^{1-q} \\ \le \left(\int_{|x|<R^2} [v(x,s) - u(x,s)]_+\,dx\right)^{1-m} + K\left(\frac{|t-s|}{\log R}\right)^{1-m}.\end{aligned} \tag{3.2.41}$$

Proof of Proposition 3.2.6. We consider a non-standard cut-off function

$$\zeta_R(x) = Z(\sigma)^k, \quad \sigma = \frac{\log r}{\log R}, \; r = |x|$$

with $Z = Z(\sigma)$ non-negative, smooth such that $Z(\sigma) = 1$ for $\sigma \le 1$, $Z(\sigma) = 0$ for $\sigma \ge 2$, and $k > 2$ to be chosen. A simple computation shows

$$\Delta\zeta_R(x) = \frac{k\, Z^{k-2}}{r^2 \log^2 R}\left[\, Z\, Z'' + (k-1)\,(Z')^2\right].$$

We now proceed in the standard way by subtracting equation (3.2.1) for u and v and multiplying by $\text{sign}_+(v-u)\,\zeta_R(x) = \text{sign}_+(\log v - \log u)\,\zeta_R(x)$. Integrating also by parts we obtain

$$\frac{d}{dt}\int [v(x,t) - u(x,t)]_+\, \zeta_R(x)\, dx \le \int [\,\log v - \log u]_+\, \Delta\zeta_R(x)\, dx.$$

We now use the inequality

$$[\,\log v - \log u]_+ \le \frac{1}{m}\left[\frac{v}{u} - 1\right]_+^m, \qquad 0 < m < 1$$

and Hölder's inequality to get

$$\left|\frac{d}{dt}\int [v-u]_+\, \zeta_R\, dx\right| \le \frac{1}{m}\, I(t)^{1-m}\left(\int [v-u]_+\, \zeta_R\, dx\right)^m \tag{3.2.42}$$

where

$$\begin{aligned} I(t)^{1-m} &= \left(\int u^{\frac{-m}{1-m}}\, \zeta_R^{\frac{-m}{1-m}}\, |\Delta\zeta_R|^{\frac{1}{1-m}}\, dx\right)^{1-m} \\ &\le \frac{k}{(\log R)^2}\left(\int_{R<|x|<R^2} u^{\frac{-m}{1-m}}\, r^{\frac{-2}{1-m}}\, |\tilde Z_m(\sigma)|^{\frac{1}{1-m}}\, dx\right)^{1-m} \end{aligned} \tag{3.2.43}$$

and we have chosen $k = 2/(1-m)$ and set $\tilde Z_m = Z\,Z'' + (k-1)\,(Z')^2$.

We next use the growth condition (3.2.4) to estimate

$$\begin{aligned} I(t)^{1-m} &\le \frac{2(2\pi)^{1-m} C(\tau_1,\tau_2)^{-m}}{1-m}\, \frac{t^{-m}}{(\log R)^2}\left(\int_R^{R^2} (\log r)^{\frac{2m}{1-m}}\, |\tilde Z_m(\sigma)|^{\frac{1}{1-m}}\, \frac{dr}{r}\right)^{1-m} \\ &= K(\tau_1,\tau_2,m)\, \frac{t^{-m}}{(\log R)^{1-m}}\left(\int \sigma^{\frac{2m}{1-m}}\, |\tilde Z_m(\sigma)|^{\frac{1}{1-m}}\, d\sigma\right)^{1-m} \end{aligned}$$

for $\tau_1 \le t \le \tau_2$. Integrating (3.2.42) with respect to time we finally obtain

$$\begin{aligned} &\left|\left(\int [v(t) - u(t)]_+\, \zeta_R\, dx\right)^{1-m} - \left(\int [v(s) - u(s)]_+\, \zeta_R\, dx\right)^{1-m}\right| \\ &\le K(\tau_1,\tau_2,m)\left(\frac{|t-s|}{\log R}\right)^{1-m}\left(\int_1^2 \sigma^{\frac{2m}{1-m}}\, |\tilde Z_m(\sigma)|^{\frac{1}{1-m}}\, d\sigma\right)^{1-m} \end{aligned}$$

showing the desired result. □

Proof of Theorem 3.2.2. We begin by showing that (3.2.4) implies uniqueness. Indeed assuming that u, v are two solutions of (3.2.1) on $\mathbb{R}^2 \times (0, T)$ satisfying (3.2.4), then by letting $R \to \infty$ and $s \to 0$ in (3.2.41), we obtain the estimate

$$\left(\int_{\mathbb{R}^2} |v(x,t) - u(x,t)|\, dx \right)^{1-m} \leq \left(\int_{\mathbb{R}^2} |v(x,0) - u(x,0)|\, dx \right)^{1-m}$$

holding for any $0 < t < T$ and $0 < m < 1$. Hence, uniqueness readily follows.

It remains to show that the maximal solution which was constructed in Theorem 3.2.1 satisfies the growth condition (3.2.4). This easily follows by comparing u with explicit solutions to equation $u_t = \Delta \log u$ as the proof of the following proposition shows.

Proposition 3.2.7. *Assume that u is as in Theorem 3.2.1. Then, for any $0 < t_0 < T$ and $R > 1$, there exists a constant $C > 0$ such that*

$$u(x,t) \geq \frac{C\, t}{|x|^2 \log^2 |x|} \quad \textit{for all } |x| \geq R,\ 0 \leq t \leq t_0. \tag{3.2.44}$$

Proof of Proposition 3.2.7. As in the proof of Theorem 3.2.1, for $\varepsilon > 0$, we let u_ε denote the solution of problem (3.2.1) with initial data $u_0 + \varepsilon$. We first remark that by the Aronson–Bénilan inequality $u_{\varepsilon t} \leq u_\varepsilon / t$, we have

$$u_\varepsilon(x,t) \geq \frac{t}{t_0}\, u_\varepsilon(x,t_0) \geq \frac{t}{t_0}\, \varepsilon \quad \text{for all } |x| \geq R,\ \ 0 < t \leq t_0$$

so that u_ε satisfies (3.2.4) in the region $Q_{R,t_0} = \{(x,t) : |x| > R,\ 0 < t \leq t_0\}$, with a constant depending only on ε, R and t_0. We shall next compare u_ε with the explicit solution

$$\psi(x,t) = \frac{2t}{|x|^2 \log^2\left(\frac{|x|}{R'}\right)}$$

in Q_{R,t_0} with R' suitably chosen. Again from the Aronson–Bénilan inequality we have

$$u_\varepsilon(x,t) \geq u(x,t) \geq \frac{t}{t_0}\, u(x,t_0) \geq \frac{t}{t_0}\, \delta$$

with $\delta := u(x, t_0) > 0$, because $t_0 < T$ (see the proof of Theorem 3.2.1). Hence, the boundary comparison $\psi(x,t) \leq u_\varepsilon(x,t)$ for $|x| = R$ holds provided we choose $R' = R'(R, t_0, \delta)$ so that $\psi(x,t) \leq t\,\delta/t_0$, for $|x| = R$ and $0 < t \leq t_0$. By a comparison argument similar to that of Proposition 3.2.6 we conclude that $u_\varepsilon(x,t) \geq \psi(x,t)$ on Q_{R,t_0}, so letting $\varepsilon \to 0$ we finally obtain (3.2.44). □

Some comments. It is shown by Esteban et al. [117] that the maximal solution is also characterized by the flux condition

$$\lim_{r \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{r\, u_r(r,\theta,t)}{u(r,\theta,t)}\, d\theta = -2.$$

In addition it is shown in [117] that the maximal solution is bounded, due to an $L^1 - L^\infty$ regularizing effect. An $L^p - L^\infty$ regularizing effect, with $p > 1$, has been previously shown by Davis et al. [62].

It is shown by Vazquez et al. [131] that in the radially symmetric case the function $\phi(t)$ in (3.2.6) is related to the *flux* of the solution $u = u(r,t)$ at infinity, namely

$$\lim_{r \to +\infty} \frac{r\, u_r(r,t)}{u(r,t)} = -\Phi(t) \tag{3.2.45}$$

with $\Phi(t) = \int_0^t \phi(s)\, ds$. In addition, when $\Phi(t) \in L^\infty(0,T)$ the solution is characterized uniquely by (3.3.6). In the non-radial case the uniqueness of the intermediate solutions under (3.2.5) or a suitable flux condition is an open question.

Existence of solutions of (3.2.1) under the additional assumption that $u_0 \in L^p(\mathbb{R}^2)$ has been also shown by Hui in [93], where an $L^p - L^\infty$ regularity result is also obtained.

Existence of global in time solutions of (3.2.1) with $\int_{\mathbb{R}^2} u_0\, dx = \infty$ and satisfying certain additional geometric assumptions was first shown by Wu [133], [134] where it was also established the convergence of the solution as $t \to \infty$. A more general convergence at $t \to \infty$, when $\int_{\mathbb{R}^2} u_0\, dx = \infty$, was recently established by Hsu [87], [84].

It has been shown by Hui [91] that the Cauchy problem (3.2.1) in dimension $n = 2$ and initial data the Dirac mass admits no solution. One may ask: what is the largest class of admissible initial data of (3.2.1)? This question is related to the results of Chasseigne and Vazquez in [36].

Part II. The Cauchy problem (3.2.1) in dimensions $n \geq 3$. We shall now discuss the solvability of the Cauchy problem (3.2.1) in the sub-critical case of dimension $n \geq 3$. The initial data u_0 will be assumed to be non-negative and locally integrable. The solvability of (3.2.1) is well understood in the radially symmetric case. However there are striking differences between the radial and the non-radial situation. These differences do not appear in dimensions $n = 1, 2$ corresponding to the supercritical and critical cases, respectively.

Define for $r > 0$, the operator $N(f)(r)$ by

$$N(f)(r) = \int_0^r \frac{1}{\omega_N s^{n-1}} \int_{B_s(0)} f(x)\, dx \tag{3.2.46}$$

where ω_N denotes the surface area of the unit sphere in $\mathbb{R}^n$. The following results have been obtained by Daskalopoulos and del Pino in [54].

Theorem 3.2.8. *Let $u_0 \geq 0$ and locally integrable on $\mathbb{R}^n$, with $n \geq 3$. If there exists a number $\tau < T$ such that*

$$\int_0^\infty \exp\left(-\frac{N(u_0)(r)}{\tau}\right) r dr = +\infty \tag{3.2.47}$$

then there is no solution of problem (3.2.1) defined on $\mathbb{R}^n \times (0,T)$.

The converse assertion is valid under the additional assumption of *radial symmetry* for the initial data, as stated next.

Theorem 3.2.9. *Assume that* $u_0 \geq 0$ *is radially symmetric and locally integrable on* $\mathbb{R}^n$*, with* $n \geq 3$*. Then there exists a solution of problem* (3.2.1) *on* $\mathbb{R}^n \times (0, T)$ *if and only if*

$$\int_0^\infty \exp\left(-\frac{N(u_0)(r)}{\tau}\right) r\,dr < +\infty \tag{3.2.48}$$

for all $0 < \tau < T$.

Let us define the *maximal time of existence* for u_0 to be the number

$$T_{\max}(u_0) = \sup\{T > 0 : \text{there exists a solution of (3.2.1)}\},$$

where this number is understood to be zero if the supremum is taken over the empty set. Then Theorem 3.2.8 may be expressed as

$$T_{\max}(u_0) \leq \sup\left\{T > 0 : \int_0^\infty \exp\left(-\tfrac{N(u_0)(r)}{T}\right) r\,dr < +\infty\right\}.$$

On the other hand if u_0 is radially symmetric, Theorem 3.2.9 states that the above inequality holds with equality. It is worth mentioning that in this case the characterization is as satisfactory as the classical result for the heat equation $u_t = \Delta u$ in $\mathbb{R}^n$ with $u(0, x) = u_0(x)$, which reads

$$T_{\max}(u_0) = \sup\left\{T > 0 : \int \exp\left(-\tfrac{|x|^2}{4T}\right) u_0(x)\,dx < +\infty\right\}.$$

Let us compare this situation with that of the two-dimensional case. If $n = 2$ then there exists a solution to (3.2.1) in $\mathbb{R}^2 \times (0, T)$ if and only if (3.2.2) holds. It is easy to check that for u_0 in $L^1(R^2)$ one has

$$N(u_0)(r) = \frac{1}{4\pi}\left(\int_{R^2} u_0(x)\,dx\right) \log r + o(\log r)$$

and hence condition (3.2.48) holds for all $0 < \tau < T$ if and only if (3.2.2) holds. Since the latter condition guarantees solvability no matter whether or not u_0 is radial, one may expect the same to be true in higher dimensions without this extra assumption. However, surprisingly, this is not the case in higher dimensions as we shall exhibit by an explicit counterexample in the following theorem.

Theorem 3.2.10. *Assume* $n \geq 3$*. Then, given any numbers* $T_0 > 0$ *and* ε, $0 < \varepsilon < T_0$*, there exists an* $u_0 \in L^1_{\text{loc}}(R^n)$ *such that*

$$\int_0^\infty \exp\left(-\frac{N(u_0)(r)}{\tau}\right) r\,dr < +\infty.$$

for all $0 < \tau < T_0$*, but for which there is no solution of* (3.2.1) *for* $T = \varepsilon$.

We should mention that this type of striking differences between the radial and non-radial situations had already been observed in [54] for a problem of the form $u_t + \Delta u^{-m} = 0$ with $m > 0$.

We shall proceed with the proofs of the above results. The basic idea is to establish a workable link between the solvability of problem (3.2.1) and that of certain elliptic equations of the form

$$- \Delta v + K(|x|)e^v = 0 \tag{3.2.49}$$

for a decaying potential K. It is known for instance, under some additional conditions, that solvability of this problem implies that $\int_0^\infty K(r)\, r dr$ is finite. The relationship between this condition and solvability of the elliptic problem (3.2.49) in dimensions greater than two has been studied in the literature, we refer the reader to [39], [40], [41], [42] and references therein. This link is the underlying mechanism in the proofs of Theorems 3.2.8 and 3.2.9. The parabolic counterexample in the proof of Theorem 3.2.10 relies on finding counterexample to solvability to a problem of the form

$$-\Delta v + e^v = u_0$$

when $u_0 = u_0(x)$ is a function large in average in large balls, yet leaving a big empty space. Non-radiality is essential in this construction, as is the fact that the dimension is greater than two.

Proof of Theorem 3.2.8. We recall that if $N(f)$ is the operator defined by (3.2.46), then for any number $r > 0$ and any function $f \in C^2(\mathbb{R}^n)$ we have

$$N(\Delta f)(r) = \bar{f}(r) - f(0)$$

where $\bar{f}(r)$ denotes the spherical average of f on the sphere of radius r centered at the origin, namely

$$\bar{f}(r) = \frac{1}{\omega_N r^{N-1}} \int_{\partial B_r(0)} f(\sigma) d\sigma.$$

The proof of Theorem 3.2.8 will be based on the following elliptic non-existence result.

Lemma 3.2.11. *Assume that $n \geq 3$ and consider the elliptic problem*

$$- \Delta v + K(r)\, e^v = 0 \quad \text{on } \mathbb{R}^n \tag{3.2.50}$$

with $K(r) > 0$ locally bounded, radially symmetric and strictly decreasing. Then, if problem (3.2.50) *admits a solution one must have*

$$\int_0^\infty K(r)\, r dr < \infty.$$

Proof of Lemma 3.2.11. In the case where the function $K(r)$ satisfies the bound

$$K(r) \leq \frac{C}{r^2} \quad \text{for all } r \geq R_0$$

for some constants $C > 0$ and $R_0 > 0$, the proof of this result is well known in the literature and we shall omit it here, referring the reader to [39]. Hence, we shall assume that this bound is not satisfied, namely there exists an increasing sequence $R_n \to +\infty$ such that

$$K(R_n)\, R_n^2 \to +\infty \quad \text{as } n \to \infty.$$

We shall then show that the maximal solution v to (3.2.50) must have $v(0) = -\infty$, which is impossible. Indeed, for $R > 0$, let v_R be the unique solution to the problem

$$\left.\begin{aligned} -\Delta v_R + K(R)\, e^{v_R} &= 0 && \text{in } B_R(0),\\ v_R &= +\infty && \text{on } \partial B_R(0). \end{aligned}\right\} \tag{3.2.51}$$

Since

$$K(r) \geq K(R) \quad \text{for all } r \leq R$$

the maximal solution v to the equation (3.2.50) satisfies

$$-\Delta v + K(R)\, e^{v} \leq 0$$

and hence by the maximum principle one must have

$$v \leq v_R \quad \text{in } B_R(0).$$

We shall now show that for the given sequence $R_n \to \infty$ we have

$$\lim_{n\to\infty} v_{R_n}(0) = -\infty$$

which would imply that $v(0) = -\infty$ as desired. To this end, we observe that if w is the solution to the problem

$$\begin{aligned} -\Delta w + e^{w} &= 0 && \text{in } B_1(0),\\ w &= +\infty && \text{on } \partial B_1(0), \end{aligned}$$

then by uniqueness,

$$v_R(r) = w\left(\frac{r}{R}\right) - \log(K(R)\, R^2).$$

In particular

$$v_R(0) = w(0) - \log(K(R)\, R^2)$$

and hence $v_{R_n}(0)$ tends to $-\infty$ since by assumption $K(R_n)\, R_n^2$ tends to $+\infty$. □

An immediate consequence of the previous result is:

Corollary 3.2.12. *Assume that there exists a non-negative, radial, smooth subsolution w of the problem*

$$-\Delta w + K(r)\, e^{w} \leq 0 \quad \text{on } \mathbb{R}^n$$

where $K(r) > 0$, locally bounded, radially symmetric and strictly decreasing. Then

$$\int_0^\infty K(r)\, r\, dr < +\infty.$$

We are now ready to finish the proof of Theorem 3.2.8. We shall proceed by contradiction. Assume that there exists a solution u of problem (3.2.1) defined on $\mathbb{R}^n \times (0, T)$. Without loss of generality we may assume that u is the maximal solution. Then the Harnack estimate established in [124] shows that u is bounded away from zero for $t > 0$ and therefore smooth by the classical parabolic regularity theory.

We shall show that for any number τ in $0 < \tau < T$ we must have

$$\int_0^\infty \exp\left(- \frac{N(u_0)(r)}{\tau} \right) r dr < +\infty$$

contradicting our assumption.

Indeed, fix a number τ in $0 < \tau < T$ and a number T_0 in $\tau < T_0 < T$. Then the solution u is defined on $\mathbb{R}^n \times [0, T_0]$. Applying the operator N to the equation

$$\frac{\partial u}{\partial t} = \Delta \log u$$

we obtain that for all $r > 0$ and $0 < t \le T_0$ we have

$$\frac{\partial N(u)(r,t)}{\partial t} = \overline{\log u}(r,t) - \log u(0,t).$$

However by Jensen's inequality one has

$$\overline{\log u}(r,t) \le \log \bar{u}(r,t)$$

and therefore, since

$$N(u)(r,t) = N(\bar{u})(r,t)$$

we obtain

$$\frac{\partial N(\bar{u})(r,t)}{\partial t} \le \log \bar{u}(r,t) - \log u(0,t).$$

Integrating in time on the interval $[0, t]$, with $0 < t \le T_0$ we get

$$N(\bar{u})(r,t) - N(f)(r) \le \int_0^t \log \bar{u}(r,s)\, ds - \int_0^t \log u(0,s)\, ds$$

and since

$$\left| \int_0^t \log \bar{u}(0,s)\, ds \right| \le C$$

for all $0 < t \le T_0$ we write

$$N(\bar{u})(r,t) - N(f)(r) \le \int_0^t \log \bar{u}(r,s)\, ds + C.$$

Hence, for any positive integer $m \ge 1$ we have

$$N(\bar{u})(r,t) - N(f)(r) \le \int_0^t \log(s^m\, \bar{u}(r,s))\, ds - m \int_0^t \log s\, ds + C.$$

Using Jensen's inequality we conclude that

$$N(\bar{u})(r,t) - N(f)(r) \le t \,\log\left(\frac{1}{t}\int_0^t s^m\,\bar{u}(r,s)\,ds\right) + C,$$

which leads to the estimate

$$N(\bar{u})(r,t) - N(f)(r) \le t \,\log\left(\int_0^{T_0} s^m\,\bar{u}(r,s)\,ds\right) + C$$

for some positive constant C. Multiplying by t^m and integrating in time on the interval $0 \le t \le T_0$, we obtain

$$\begin{aligned} N\left(\int_0^{T_0} t^m \bar{u}\,dt\right)(r) - \frac{T_0^{m+1}}{m+1} N(u_0)(r) \\ \le \frac{T_0^{m+2}}{m+2}\,\log\left(\int_0^{T_0} t^m \bar{u}(r,t)\,dt\right) + C\,T_0^{m+1}. \end{aligned}$$

Therefore, setting

$$w(r) = N\left(\int_0^{T_0} t^m\,\bar{u}\,dt\right)(r)$$

we find that w satisfies the inequality

$$\frac{m+2}{T_0^{m+2}}\,w(r) - \frac{m+2}{m+1}\,\frac{N(u_0)(r)}{T_0} \le \log(\Delta w) + C.$$

Applying the exponential function on both sides of the above equation we obtain that w satisfies

$$-\Delta w + K\,e^{c\,w} \le 0 \quad \text{on } \mathbb{R}^n$$

with

$$K(r) = C\,\exp\left(-\frac{m+2}{m+1}\,\frac{N(u_0)(r)}{T_0}\right).$$

and

$$c = \frac{m+2}{T_0^{m+2}}.$$

Hence, the function $w = c\,w$ satisfies

$$-\Delta w + c\,K\,e^{w} \le 0 \quad \text{on } \mathbb{R}^n,$$

It follows from Corollary 3.2.12 that one must have

$$\int_0^\infty c\,K(r)\,r dr < \infty$$

which implies that

$$\int_0^\infty \exp\left(- \frac{m+2}{m+1} \frac{N(u_0)(r)}{T_0} \right) r\, dr < \infty.$$

However, if m is chosen large enough so that

$$\frac{m+1}{m+2} T_0 > \tau$$

the above condition would imply that

$$\int_0^\infty \exp\left(- \frac{N(u_0)(r)}{\tau} \right) r\, dr < \infty$$

contradicting the assumption (3.2.47). Therefore there is no solution of problem (3.2.1) on $\mathbb{R}^n \times (0, T)$. □

Before we present the proof of Theorem 3.2.9 we give the following proposition that allows us to connect the parabolic problem to the corresponding elliptic problem

$$-\Delta v + e^v = f \quad \text{on } \mathbb{R}^n. \tag{3.2.52}$$

Proposition 3.2.13. *Let u be a solution of equation*

$$\frac{\partial u}{\partial t} = \Delta \log u$$

in the cylinder $\Omega \times [0, T]$, where $\Omega \subset \mathbb{R}^n$, with initial data $u(x, 0) = f$. Assume in addition that u is strictly positive and smooth on $\Omega \times (0, T]$ and satisfies the Aronson–Bénilan inequality

$$\frac{\partial u}{\partial t} \le \frac{1}{t} u \quad \text{for } 0 < t \le T. \tag{3.2.53}$$

Then the function

$$\Phi(x, t) = \frac{1}{t} \int_0^t \log u(x, s)\, ds$$

defined on $\Omega \times (0, T]$ satisfies the differential inequality

$$-\Delta \Phi + \frac{1}{t} e^{\Phi + 1} \ge \frac{1}{t} f. \tag{3.2.54}$$

Proof. First notice that the Aronson–Bénilan inequality implies that for $0 < s \le t < T$, one has

$$\log u(x, s) \ge \log u(x, t) + \log s - \log t$$

and therefore

$$\Phi(x, t) = \frac{1}{t} \int_0^t \log u(x, s)\, ds \ge \log u(x, t) - 1$$

which implies that

$$u(x,t) \le e^{\Phi+1}.$$

On the other hand

$$\Delta\Phi(x,t) = \frac{1}{t}\int_0^t \Delta \log u(x,s)\,ds = \frac{1}{t}\int_0^t \frac{\partial u(x,s)}{\partial s}\,ds = \frac{u(x,t)}{t} - \frac{f(x)}{t}.$$

Combining the above relations, we obtain

$$\Delta\Phi \le \frac{1}{t}e^{\Phi+1} - \frac{1}{t}f$$

from which (3.2.54) immediately follows. □

The proof of Theorem 3.2.9 will follow from the Proposition 3.2.13 and the following elliptic result.

Proposition 3.2.14. *Assume that $K(r)$ is non-negative, locally bounded, radially symmetric and satisfies*

$$\int_0^\infty K(r)\,r\,dr < \infty.$$

Then, for every number α, there exists a radially symmetric solution of the elliptic equation

$$-\Delta v + K(r)\,e^v = 0 \quad on\ \mathbb{R}^n \tag{3.2.55}$$

such that

$$\lim_{r\to\infty} v(r) = \alpha. \tag{3.2.56}$$

Proof. We shall only sketch the proof of this elliptic result, referring the reader to [39] and the references therein for the details. It is enough to construct a subsolution and a supersolution of the problem (3.2.55) that satisfy (3.2.56).

Since $K(r) > 0$ the constant function $v(r) = \alpha$ is certainly a supersolution of (3.2.52). To construct a subsolution we look for a radial function $v(r)$ of the form

$$v(r) = N(e^\alpha K)(r) + C$$

for an appropriate constant C to be determined, where

$$N(K)(r) = \int_0^r \frac{ds}{s^{N-1}}\int_0^s K(\rho)\,\rho^{N-1}\,d\rho = \int_0^r K(\rho)\rho^{N-1}\left(\int_\rho^r \frac{ds}{s^{N-1}}\right)d\rho.$$

It is clear that the function $v(r)$ is increasing since $K(r) \ge 0$. Moreover $v(r)$ is bounded above, since

$$N(K)(r) = \int_0^r \frac{K(\rho)}{N-2}\,\rho^{N-1}\left(\int_\rho^r \left(1 - \left(\frac{\rho}{s}\right)^{N-1}\right)ds\right)d\rho \le \int_0^\infty \frac{K(\rho)}{N-2}\,\rho\,d\rho.$$

Therefore the limit

$$L := \lim_{r\to\infty} N(e^{\alpha} K)(r)$$

exists and it is finite. Hence, choosing the constant C so that

$$\lim_{r\to\infty} v(r) = L + C = \alpha$$

we see that v is the desired subsolution. □

Proof of Theorem 3.2.9. The only-if assertion is already contained in Theorem 3.2.8. Hence, we restrict our attention to constructing a solution u of the initial value problem (3.2.1) under the assumption of Theorem 3.2.9. The solution we are seeking will be constructed as the limit as $R \to \infty$ of the radial solutions u_R of the boundary value problems

$$\left.\begin{aligned} \partial u/\partial t &= \Delta \log u && \text{in } B_R \times [0,\infty),\\ u(R,t) &= +\infty && t \in (0,\infty),\\ u(r,0) &= f(r) && x \in B_R. \end{aligned}\right\} \tag{3.2.57}$$

Here B_R denotes the ball of radius R centered at the origin. Be begin by showing that problem (3.2.57) admits a solution u_R. Indeed, let w_R denote the unique solution of the elliptic problem

$$\left.\begin{aligned} -\Delta \log w + w = 0, &\qquad \text{in } B_R,\\ w(R) = +\infty. & \end{aligned}\right\} \tag{3.2.58}$$

It is well known that $w_R > 0$ and smooth for $r < R$. A direct computation shows that for any constant C the function

$$W_R(r,t) = (t + C)\, w_R(r)$$

solves the initial boundary value problem (3.2.57) with initial data $W_R(r,0) = C\, w_R(r)$. Hence, the solution u_R to the problem (3.2.57) can be easily constructed via a standard approximation argument, since it will satisfy the bounds

$$t\, w_R(r) \le u_R(r,t) \le (t + C_R)\, w_R(r) \quad \text{on } r \le R$$

with

$$C_R = \|f\|_{L^\infty(B_R)}.$$

Moreover, u_R will be strictly positive and smooth when $r < R$ and $t > 0$ and therefore it will satisfy the Aronson–Benilan inequality (3.2.53), see [7]. By the maximum principle we see that the sequence u_R is decreasing as $R \to \infty$ and therefore the limit

$$u(r,t) = \lim_{R\to\infty} u_R(r,t)$$

exists and is finite for all $r \ge 0$, $0 < t \le \infty$.

Our goal is to show that u is actually a solution of the problem (3.2.1) on $\mathbb{R}^n \times (0,T)$. It is clear the u is non-negative and locally bounded on $\mathbb{R}^n \times (0,T)$. Using

Propositions 3.2.13 and 3.2.14 we shall show that for all $R > 0$ and $0 \le \tau < T$ we have

$$\log u(r,\tau) \ge -\frac{C(\rho,T)}{T-\tau} + \log\tau \quad \text{for } r < \rho \tag{3.2.59}$$

where $C(\rho,\tau,T)$ is a constant independent of r and t. We shall actually show that (3.2.59) is satisfied by each of the approximate solutions u_R, when $\rho < R$. To this end, set

$$\Phi_R(r) = \frac{1}{T}\int_0^T \log u_R(r,s)\,ds.$$

Then, by Proposition 3.2.13, Φ_R satisfies

$$-\Delta\Phi_R + \frac{1}{T}e^{\Phi_R+1} \ge \frac{u_0}{T} \quad \text{on } r < R,$$

and hence the function

$$\Psi_R(r) = \Phi_R(r) + \frac{N(u_0)(r)}{T}$$

satisfies

$$-\Delta\Psi_R + K(r)\,e^{\Psi_R} \ge 0 \quad \text{on } r < R$$

with

$$K(r) = \frac{e}{T}\exp\left(-\frac{N(u_0)(r)}{T}\right) \tag{3.2.60}$$

and boundary condition

$$\Psi_R = +\infty \quad \text{on } r = R.$$

Let $v = v(r)$ be the solution of the (3.2.50) with $K(r)$ given by (3.2.60) and such that

$$\lim_{r\to\infty} v(r) = 0.$$

Such a solution exists because of the assumption

$$\int_0^\infty \exp\left(-\frac{N(u_0)(r)}{T}\right) r\,dr < \infty.$$

Then, by the maximum principle we must have

$$\psi_R \ge v \quad \text{on } r < R$$

which implies that

$$\phi_R(r) = \frac{1}{T}\int_0^T \log u_R(r,t)\,dt \ge v(r) - \frac{N(u_0)(r)}{T}$$

for $r < R$. Now fix a number τ in $0 < \tau < T$. Then, by the previous estimate

$$\frac{1}{T}\int_\tau^T \log u_R(r,t)\,dt \ge -\frac{1}{T}\int_0^\tau \log u_R(r,t)\,dt + v(r) - \frac{N(u_0)(r)}{T}.$$

The function u_R satisfies the bound

$$u_R(r,t) \le C(\rho, T)$$

for all r in $r \le \rho < R$ and t on $0 \le t \le T$. This is because the initial data u_0 is assumed to be locally bounded. Therefore, since the function v is locally bounded, we conclude that

$$\int_\tau^T \log u_R(r,t)\, dt \ge -C(\rho, T) \quad \text{for } r < R$$

with the constant C depending only on ρ and T. On the other hand the Aronson–Bénilan inequality implies that

$$\log u_R(r,\tau) \ge \log u_R(r,t) + \log \tau - \log t$$

for all t in $\tau \le t \le T$. Integrating in time, we obtain that

$$(T-\tau)\, \log u_R(r,\tau) \ge \int_\tau^T \log u_R(r,t)\, dt + (T-\tau)\, \log \tau - C(T)$$

where $C(T)$ denotes a constant depending only on T. Combining the above we get

$$\log u_R(r,\tau) \ge \frac{C(\rho,T)}{T-\tau} + \log \tau \tag{3.2.61}$$

for all $r \le \rho < R$ and $0 < \tau < T$, where $C(\rho, T)$ is independent of R and τ. Letting $R \to \infty$ in (3.2.61) we obtain (3.2.59). Having shown (3.2.59) it is now easy to see using known arguments that u is actually a solution of the initial value problem (3.2.1) on $\mathbb{R}^n \times (0,T)$ which is smooth on $\mathbb{R}^n \times (0,T)$. This finishes the proof of Theorem 3.2.9. □

The proof of Theorem 3.2.10 will be based on the following construction. Let us consider the equation

$$-\Delta u + e^u = u_0(x) \quad \text{on } \mathbb{R}^n. \tag{3.2.62}$$

Proposition 3.2.15. *Assume $n \ge 3$. Then, given $M > 0$, there exists u_0, locally bounded in $\mathbb{R}^n$ such that*

$$\liminf_{R\to+\infty} \frac{1}{R^{N-2}} \int_{B_R} u_0(x)dx > M \tag{3.2.63}$$

but for which problem (3.2.62) does not have any subsolution, that is no $u(x)$ locally integrable exists for which

$$-\Delta u + e^u \le u_0(x) \tag{3.2.64}$$

in the distributional sense.

Proof. We shall actually provide an explicit u_0 for which this proposition holds. To start the construction, we consider a continuous, non-negative function $\delta(x)$ whose support lies in the ball $B(0,1)$. Let also $a > 1$ be fixed. We consider the following function

$$f_0(x) = \sum_{i=-\infty}^{\infty} a^{-2i}\delta(a^{-i}x - \boldsymbol{e}), \tag{3.2.65}$$

where $\boldsymbol{e} = (1, 0, \ldots, 0)$. We observe that this series is convergent at every point except at $x = 0$ and it defines a locally integrable function. Moreover, it is straightforward to check that

$$\int_{B_R(0)} f_0(x)dx \geq \left(\int \delta(y)dy\right) \frac{a}{a-1} R^{N-2} + O(1),$$

hence

$$\liminf_{R\to+\infty} \frac{1}{R^{N-2}} \int_{B_R} f_0(x)dx \geq \left(\int \delta(y)dy\right) \frac{a}{a-1}. \tag{3.2.66}$$

An important step in the proof will be the construction of a special family of supersolutions to equation (3.2.62). To this end we set

$$w_n(x) = \sum_{i=-\infty}^{n} \psi(a^{-i}x - \boldsymbol{e}) \tag{3.2.67}$$

where ψ denotes the Newtonian potential of δ, namely

$$\psi(z) = c_N \int \frac{\delta(y)}{|z-y|^{N-2}} dy.$$

Next we shall evaluate the function $w_n(x)$ for an n to be fixed later, along the following sequence: fix a vector $\boldsymbol{v}$ with $\boldsymbol{v} \cdot \boldsymbol{e} > 0$ and define $x_k = -a^{-k}\boldsymbol{v}$ with $k = 0, 1, 2, \ldots$. Then

$$\begin{aligned} w_n(x_k) &= \sum_{i=-\infty}^{n} \psi(a^{-i-k}\boldsymbol{v} + \boldsymbol{e}) \\ &= \sum_{j=1}^{k} \psi(a^{-j}\boldsymbol{v} + \boldsymbol{e}) + \sum_{j=0}^{\infty} \psi(a^{j}\boldsymbol{v} + \boldsymbol{e}) + \sum_{i=1}^{n} \psi(a^{-i-k}\boldsymbol{v} + \boldsymbol{e}). \end{aligned}$$

It follows that

$$w_n(x_k) \leq C(n) + k \int \frac{\delta(y)dy}{|y + \boldsymbol{e}|^{N-2}}.$$

We make the following requirement

$$w_n(x_k) - 2\log|x_k| \to -\infty \quad \text{as } k \to \infty. \tag{3.2.68}$$

For (3.2.68) to hold, it is thus sufficient that

$$\int \frac{\delta(y)dy}{|y + \boldsymbol{e}|^{N-2}} < \log a. \tag{3.2.69}$$

On the other hand, we shall also require that

$$\left(\int \delta(y)dy\right)\frac{a}{a-1} > M \tag{3.2.70}$$

so that, using (3.2.66), we also have the validity of the inequality (3.2.63) for f_0. It is clear that assuming that δ_0 is supported in the unit ball and satisfies

$$\int \delta_0(y)dy = 1, \quad \int \frac{\delta_0(y)dy}{|y+\boldsymbol{e}|^{N-2}} = 2,$$

then, setting $\delta = 2M\delta_0$ and choosing $a = e^{3M}$, one gets that properties (3.2.69) and (3.2.70) are simultaneously satisfied.

Next we claim that if n in the definition of w_n is taken sufficiently large then w_n satisfies

$$-\Delta w_n + e^{w_n} \geq f_0. \tag{3.2.71}$$

In fact, we have that in the distributional sense

$$-\Delta w_n(x) = \sum_{i=-\infty}^{n} a^{-2i}\delta(a^{-i}x - \boldsymbol{e}).$$

Thus,

$$-\Delta w_n(x) + e^{w_n(x)} \geq f_0(x) + \Big(1 - \sum_{i=n}^{\infty} a^{-2i}\delta(a^{-i}x - \boldsymbol{e})\Big).$$

But since δ is a uniformly bounded function and $a > 1$, we can choose a sufficiently large n so that the second term in the right hand side of the above inequality is positive and then (3.2.71) holds. Let us now define

$$u_0(x) = \min\{f_0(x), A\} \tag{3.2.72}$$

where A is chosen so large that (3.2.63) still holds. It follows that for this choice of n, $w_n(x)$ is indeed a super-solution of problem (3.2.62) for this u_0.

Let us choose a, $\delta(x)$ and n so that w_n given by (3.2.67) is a supersolution of (3.2.62) for u_0 given by (3.2.65).

Now, for each positive integer k, we consider the function

$$v_k(x) = w_n(a^{-k}x) - 2k\log a.$$

Then, observing that $a^{-2k}f_0(a^{-k}x) = f_0(x)$, we obtain

$$-\Delta v_k + e^{v_k} \geq f_0(x),$$

in the distributional sense, so that v_k is still a supersolution of (3.2.62). Let us now consider the following function

$$u_k(x) = v_k(x) + \frac{1}{a^k - |x|} + C.$$

Then

$$
\begin{aligned}
&-\Delta u_k + e^{u_k} \\
&\quad \geq f_0(x) + \frac{1}{(a^k - |x|)^2}\left[\frac{N-1}{|x|} - \frac{1}{(a^k - |x|)}\right] + e^{v_k}\left[\exp(\frac{1}{a^k - |x|} + C) - 1\right] \\
&\quad \geq u_0(x) + \exp\left(\frac{1}{a^k - |x|} + C\right) + \frac{1}{(a^k - |x|)^3} - 1
\end{aligned}
$$

where u_0 is defined as in (3.2.72). From this expression, we see that there is a choice of $C > 0$, uniform in k, such that u_k is still a supersolution of equation (3.2.62), now in the ball $B(0, a^k)$. Let us fix such a C.

To finish the proof of Proposition 3.2.15, assume that (3.2.62) has a subsolution u defined in entire $\mathbb{R}^n$. Then since u_k is a supersolution of (3.2.62) which is equal to infinity on $\partial B(0, a_k)$, it follows that $u(x) \leq u_k(x)$ for all $x \in B(0, a_k)$. However, let $\bar{x}$ be any point with $\bar{x} \cdot \boldsymbol{e} < 0$. Then by the definition of u_k we have

$$
u_k(\bar{x}) \leq C + w_n(a^{-k}\bar{x}) - 2k \log a
$$

for some constant C independent of k. But then, since a was chosen such that relation (3.2.68) holds, then we have that $u_k(\bar{x}) \to -\infty$ as $k \to \infty$. But this implies that $u(\bar{x}) = -\infty$ for all such an $\bar{x}$. Hence $u \equiv -\infty$ on an open set, which is a contradiction. This finishes the proof of the proposition. □

After this elliptic proposition, we are ready for the proof of Theorem 3.2.10.

Proof of Theorem 3.2.10. Let us assume that u solves (3.2.1), and fix $0 < \tau < T$. Let us set

$$
v(x) = \frac{2}{\tau^2} \int_0^\tau \int_0^s \log u(x,t)\, dt ds.
$$

Then

$$
\Delta v(x) = \frac{2}{\tau^2} \int_0^\tau \int_0^s u_t(x,t)\, dt ds = \frac{2}{\tau^2} \int_0^\tau u(x,t) dt - \frac{2}{\tau} u_0(x).
$$

Now observe the following

$$
v(x) = \frac{2}{\tau^2} \int_0^\tau (\tau - t) \log u(x,t)\, dt ds \leq \log\left(\frac{2}{\tau^2} \int_0^\tau (\tau - t) u(x,t)\, dt\right).
$$

Hence

$$
e^{v(x)} \leq \frac{2}{\tau} \int_0^\tau u(x,t)\, dt
$$

and therefore

$$
-\tau\, \Delta v(x) + e^{v(x)} \leq 2u_0(x).
$$

After a scaling, Theorem 3.2.10 reduces to the following assertion: Given $M > 0$, there exists a u_0 for which $\liminf R^{2-N} \int_{B_R} f > M$ but such that no solution of (3.2.1) exists for this u_0 and $T = 3$.

Then let us consider $M > 0$ and let us choose a u_0 such that Proposition 3.2.15 holds with f in the place of u_0 in (3.2.64). Assume that (3.2.1) has a solution for this u_0 and $T = 3$. Then defining $v(x)$ as above, for $\tau = 1$, we obtain that v satisfies distributionally

$$-\Delta v(x) + e^{v(x)} \leq 2\,u_0(x).$$

But Proposition 3.2.15 shows that such a v cannot exist, a contradiction that concludes the proof of the theorem. □

Comments. The uniqueness question for the Cauchy problem (3.2.1) in dimensions $n \geq 3$ is open. The non-uniqueness in dimension $n = 2$ implies also non-uniqueness in dimensions $n \geq 3$. However, many questions remain open. Are radial solutions unique? What conditions determine the solution to (3.2.1) uniquely?

The existence in the Neumann problem for equation $u_t = \log u$ on $B_R \times (0, T)$ in dimensions $n \geq 2$, has been shown by Hui [93] where an a-priori L^∞ estimate for solutions to (3.2.1) with initial data $u_0 \in L^p(\mathbb{R}^n) \cup L^1(\mathbb{R}^n)$ is also obtained.

3.3 Further results and open problems

In the section we shall give a brief summary of known results regarding the quantitative and qualitative behavior of solutions to the Cauchy problem for equation $u_t = \Delta\varphi(u)$, with $\varphi \in \mathcal{F}_a$ which were not covered in detail in Sections 3.1 and 3.2 of this chapter, as they are not closely related to the main objective of this book. Most of the results which will be discussed concern the pure power case $u_t = \Delta u^m$. This equation has been extensively studied in the literature. We apologize for not mentioning all the known results. We refer the reader to the survey articles [111] and [125] for a collection of known results.

1. Time asymptotic behaviour of solutions. In this section we shall present a brief summary of the results on the asymptotic behaviour in time of solutions to the Cauchy problem

$$\left.\begin{aligned} \partial u/\partial t &= \Delta u^m && \text{in } \mathbb{R}^n \times [0, T), \\ u(x, 0) &= u_0 && \text{on } \mathbb{R}^n, \end{aligned}\right\} \tag{3.3.1}$$

in the different ranges of exponents $(n-2)/n < m < 1$, $0 < m < (n-2)/n$ and the critical case $m = (n-2)/n$.

1a. The fast diffusion case $0 < m < 1$. The asymptotic analysis for the porous medium case $m > 1$ described above can be extended to the super-critical fast diffusion case $\frac{n-2}{n} < m < 1$. The source type self-similar solutions given by (2.5.11), (2.5.12) in Chapter 2 still exist, the total mass of solutions of equation (3.3.1) is still conserved in time and finite-mass non-negative solutions converge, as $t \to \infty$, to the Barenblatt solution U_M with the same mass M as the solution u.

In the sub-critical case $0 < m < \frac{n-2}{n}$ the mass is not conserved and solutions to (3.3.1) undergo extinction in finite time [18], so that for some $T = T(u_0) < \infty$ the solution u satisfies $u(x, T) \equiv 0$. This can be readily seen by the explicit solutions

$$u^T(x,t) = (2c(T-t)_+|x|^{-2})^{1/(1-m)}, \quad c = \left(n - \frac{2}{1-m}\right) > 0.$$

In the sub-critical case, equation (3.3.1) admits a unique self-similar solution of the second kind of the form [99], [112]

$$U_*(r,t) = (T-t)^\gamma f(\eta), \quad \eta = \frac{r}{(T-t)^\beta}, \quad \gamma = \frac{1-2\beta}{1-m}$$

which is proved to be asymptotically stable as $t \to T$ [74]. It is shown by Galaktionov and Peletier in [74] and formally analyzed first by King in [99] that for a class of non-negative radially symmetric finite-mass solutions of (3.3.1), which vanish at a given time $T > 0$, their asymptotic behaviour as $t \uparrow T$ is described by U_*. This implies the decay rate of the form

$$\|u\|_\infty = C(m,n)\,(T-t)^\gamma\,(1+o(1)) \quad \text{as } t \to T$$

where $\gamma = \gamma(m,n) > 0$ is the anomalous exponent.

The elliptic critical exponent $m = \frac{n-2}{n+2}$ is special and geometrically significant, as it corresponds to the Yamabe flow, i.e. the evolution of a conformally flat metric by its scalar curvature. For detailed convergence results in this case we refer the reader to Rugang Ye [135], del Pino and Sáez [65] and Galaktionov and Peletier [74].

The long time behaviour of non-negative, finite-mass solutions of the Cauchy problem (3.3.1) in the parabolic critical case $m = \frac{n-2}{n}$ has been studied by Galaktionov, Peletier and Vazquez in [75]. It marks the transition between two completely different asymptotic behaviour types. In this case solutions exist globally in time and conserve the mass. The decay rate of solutions has been established in [75] as

$$\log\|u(\cdot,t)\|_\infty = -\kappa(n)\,\|u_0\|_1^{-2/(n-2)}\,t^{n/(n-2)}\,(1+o(1)) \quad \text{as } t \to \infty \tag{3.3.2}$$

where $\kappa(n)$ is an explicit constant depending on the dimension n. Moreover, (3.3.2) gives in the first approximation the asymptotic behaviour of $\log u$ in the inner region, which is the ball of radius

$$R(t) = \exp\left\{\kappa_0\,\|u_0\|_1^{-2/(n-2)}\,t^{n/(n-2)}\right\}, \quad \kappa_0 = \kappa/n.$$

In other words, the profile of $\log u$ in the inner region becomes flat in first approximation. On the other hand, the analysis of the outer region $|x| > R(t)$ performed in [75] gives a behaviour of the form

$$\log u(r,t) = \frac{n}{2}\{\log(n-2) + \log t - 2\log r - \log\log r\}.$$

Thus, in the logarithmic scale, the profile $u(x,t)$ has a broken shape. This behaviour was predicted and formally analyzed first by King in [99].

1b. The case $m = 0$ in dimension $n = 2$. We shall discuss in this section the asymptotic in time behavior of solutions of the Cauchy problem

$$\left.\begin{aligned} \partial u/\partial t &= \Delta \log u && \text{in } \mathbb{R}^2 \times [0, T), \\ u(x, 0) &= u_0 && \text{on } \mathbb{R}^2, \end{aligned}\right\} \tag{3.3.3}$$

with $0 < T \leq \infty$ and initial data $u_0 \geq 0$ and locally integrable. This case is geometrically significant, as it corresponds to the Ricci flow, i.e. the evolution of a conformally flat metric by its Ricci curvature.

Case 1. $u_0 \in L^1(\mathbb{R}^2)$. Then, according to the results of Section 3.2, for any $\gamma > 2$ there exists a unique solution u of (3.3.3) with $T = \int_{\mathbb{R}^2} u_0(x)\, dx/2\pi\gamma$, satisfying

$$\int_{\mathbb{R}^2} u(x, t)\, dx = \int_{\mathbb{R}^2} u_0(x)\, dx - 2\pi\gamma\, t, \quad 0 < t < T. \tag{3.3.4}$$

In addition, if

$$u_0 \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2) \quad \text{for some } p > 1,\, u_0 \not\equiv 0 \tag{3.3.5}$$

and it is radially symmetric, then by the results in [69]

$$\lim_{r\to+\infty} \frac{r\, u_r(r, t)}{u(r, t)} = -\gamma \quad \text{uniformly on } [a, b] \text{ as } r \to \infty \tag{3.3.6}$$

for any $0 < a < b < T$.

S. Y. Hsu [89], [88] studied the extinction behavior of radially symmetric solutions of (3.3.3). Assume that $\gamma > 2$ and for $u_0 \geq 0$ a radially symmetric function which satisfies (3.3.5), let u be the unique solution of (3.3.3) in $\mathbb{R}^2 \times (0, T)$, with

$$T = \frac{1}{2\pi\gamma} \int_{\mathbb{R}^2} u_0\, dx$$

which satisfies (3.3.4) and (3.3.6). S.Y. Hsu showed that there exist unique constants $\alpha > 0$, $\beta > -1/2$, $\alpha = 2\beta + 1$, depending on γ, such that the rescaled function

$$v(y, \tau) = \frac{u(y/(T - t)^\beta, t)}{(T - t)^\alpha}$$

where

$$\tau = -\log(T - t)$$

converges uniformly on compact subsets of $\mathbb{R}^2$ to $\phi_{\lambda,\beta}(y)$, for some constant $\lambda > 0$, where $\phi_{\lambda,\beta}(y) = \phi_{\lambda,\beta}(r)$, $r = |y|$ is radially symmetric and satisfies the O.D.E

$$\frac{1}{r}\left(\frac{r\phi'}{\phi}\right)' + \alpha\,\phi + \beta r\phi' = 0 \quad \text{in } (0, \infty)$$

with

$$\phi(0) = 1/\lambda, \quad \phi'(0) = 0.$$

In the case where $\gamma = 4$ the above result simply gives the asymptotics

$$u(x,t) \approx \frac{8\lambda(T-t)}{(\lambda + |x|^2)^2} \quad \text{as } t \to T$$

corresponding to the geometric result of R. Hamilton [77] and B. Chow [43] that under the Ricci flow, a two-dimensional compact surface shrinks to a sphere. The extinction behavior of *non-radial* solutions to (3.3.3) satisfying (3.3.4) with $\gamma > 0$ is still an open question.

The above results do not cover the extinction behavior of *maximal* solutions of (3.3.3) which poses a delicate question. Geometrically, maximal solutions correspond to the evolution of a complete conformally flat metric on a non-compact surface under the Ricci-flow. King [99] has formally analyzed the extinction behavior of maximal solutions u of (3.3.3) satisfying

$$\int_{\mathbb{R}^2} u(x,t)\,dx = \int_{\mathbb{R}^2} u_0(x)\,dx - 4\pi\,t$$

as $t \to T$, with $T = 1/4\pi \int_{\mathbb{R}^2} u_0(x)\,dx$. His formal asymptotics show that if u_0 is radially symmetric and compactly supported then, as $t \to T$, the maximal solution u enjoys the asymptotics

$$u(x,t) \approx \frac{2\,(T-t)^2}{T\,(r^2 + \lambda\, e^{2T/(T-t)})} \quad \text{for } r = O(e^{T/(T-t)})$$

for some constant $\lambda > 0$, while

$$u(x,t) \approx \frac{2T}{r^2\,\log^2 r} \quad \text{for } (T-t)\ln r > T.$$

The above asymptotics have not been shown rigorously. However in [59] sharp geometric estimates have been established on the "geometric width" and the "maximal curvature" of maximal solutions of (3.3.3) near their extinction time. We refer the reader to this paper for the details.

Case 2. $u_0 \notin L^1(\mathbb{R}^2)$. If the initial data u_0 is locally integrable but with $\int_{\mathbb{R}^2} u_0(x)\,dx = \infty$, then according to Theorem 3.2.1 of Section 3.2, there exists a solution u of the Cauchy problem (3.3.3) on $\mathbb{R}^2 \times [0,\infty)$. S.Y. Hsu studied the asymptotic profile of u under the assumption that the initial data u_0 satisfies the growth condition

$$\frac{1}{\beta(|x|^2 + k_1)} \le u_0 \le \frac{1}{\beta(|x|^2 + k_2)} \tag{3.3.7}$$

for some constants $\beta > 0$, $k_1 > 0$, $k_2 > 0$. Under (3.3.7) the Cauchy problem (3.3.3) admits a unique solution u on $\mathbb{R}^2 \times [0,\infty)$ which satisfies the bounds

$$\frac{1}{\beta(|x|^2 + k_1\, e^{2\beta t})} \le u(x,t) \le \frac{1}{\beta(|x|^2 + k_2\, e^{2\beta t})}. \tag{3.3.8}$$

The uniqueness under (3.3.8) is shown in [88] (see also [133]).

Assume that the initial data u_0 satisfies (3.3.7). It is shown in [87] (see also [133], [134]) that the rescaled function

$$w(x,t) = e^{2\beta t}\, u(e^{\beta t}\, x, t)$$

will converge, as $t \to \infty$, uniformly on $\mathbb{R}^2$ and also in $L^1(\mathbb{R}^n)$, to the function

$$\phi_{k_0,\beta}(x) = \frac{2}{\beta(|x|^2 + k_0)}$$

for some unique constant k_0 satisfying

$$\int_{\mathbb{R}^2} (u_0 - \phi_{\beta,k_0})\, dx = 0.$$

2. Ultra-fast diffusion. We shall give in this section a brief summary of results concerning the solvability and well posedness of the Cauchy problem

$$\left.\begin{aligned} \partial u/\partial t &= \Delta u^m \quad \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) &= u_0 \qquad \text{on } \mathbb{R}^n, \end{aligned}\right\} \tag{3.3.9}$$

in the range of exponents $m < 0$. Here $T > 0$ is a given constant and u_0 a non-negative, locally integrable function. We shall refer to equation (3.3.9) when $m < 0$ as *super-diffusion* or *ultra-fast diffusion*.

By a solution of (3.3.9) we mean a non-negative function u in $C([0,T); L^1_{\text{loc}}(\mathbb{R}^n))$ such that u^m belongs to $L^1_{\text{loc}}(\mathbb{R}^n \times [0,T))$ and which satisfies the equation in the distributional sense. Note that since $m < 0$ the assumption that u^m is locally integrable implies that u is non-zero almost everywhere in $Q_T := \mathbb{R}^n \times (0,T)$.

Equation (3.3.9) with $m < 0$ arises in a number of different contexts. A known example is that of diffusion in plasma; see Lonngren and Hirose [106], Berrymann and Holland [21]. Also superdiffusivities of this type have been proposed by de Gennes [64], [63] as a model for long-range Van der Waals interactions in thin films spreading on solid surfaces. See also Stratov [118]. Ultra-fast diffusion also appears in the study of cellular automata and interacting particle systems with self-organized criticality; see [38] and the references therein. Other physical applications are mentioned in [21] and [24].

It is natural to ask under which conditions on u_0 the initial value problem is solvable, and our purpose here is to present some recent results in this direction. It is illustrative to establish comparisons with the essentially settled existence theory for the cases $m > 1$ and $0 < m < 1$ which was discussed in detail in Chapter 2.

The first result on the solvability of (3.3.9) when $m < 0$ was established by Vázquez [124]. It was shown in [124] that there is no solution of (3.3.9) with $m < 0$ and $n \geq 2$ and initial data $u_0 \in L^1(R^n)$. This is because the high diffusion takes all the mass to infinity in no time, therefore yielding non-existence of a local solution for the Cauchy problem. On the other hand, there exist solutions to (3.3.9) which decay at infinity (at a non-integrable rate), for instance

$$v^T(x,t) = (2\alpha(T-t)_+|x|^{-2})^{1/(1-m)} \tag{3.3.10}$$

with

$$\alpha = (n - 2/(1-m)) > 0.$$

It is natural to ask: what is the fastest possible decay allowed on the initial data u_0 if there is a local solution (i.e. not instant vanishing)? It turns out that a notable symmetry between the cases $m < 0$ and $m > 1$ appears. More precisely the following result was shown in [52].

Theorem 3.3.1. *Assume that there is a solution to* (3.3.9) *with* $m < 0$*. Then the initial data* u_0 *must satisfy the growth condition*

$$\limsup_{R\to\infty} \frac{1}{R^{n-2/(1-m)}} \int_{B_R} u_0\,dx \geq C^* T^{1/(1-m)} \tag{3.3.11}$$

with C^* *the precise constant*

$$C^* = \left[2\left(n - \frac{2}{1-m}\right)\right]^{1/(1-m)} \omega_n. \tag{3.3.12}$$

Here ω_n *denotes the surface area of the unit sphere.*

In other words, if the limit above is strictly less than $C^*T^{1/(1-m)}$, then any local solution must cease to exist by vanishing before time T. Note that this result is the analogue of condition (2.3.2) in Chapter 2 for the slow diffusion case $m > 1$. It is actually optimal, since for the explicit solution v^T in (3.3.10) which vanishes exactly at time T one has

$$C^*T^{1/(1-m)} = \lim_{R\to\infty} \frac{1}{R^{n-2/(1-m)}} \int_{B_R} v^T(x,0)\,dx.$$

It is tempting to guess that a condition of the form (3.3.11), possibly replacing the lim sup by lim inf, is sufficient for existence when $m < 0$. The answer is affirmative in the radially symmetric case. In fact the following result was shown in [53]

Theorem 3.3.2. *Assume that* $u_0(x) \geq g(|x|)$*, where* g *is radially symmetric and*

$$\liminf_{R\to\infty} \frac{1}{R^{N-2/(1-m)}} \int_{B_R} g(|x|)dx \geq C^* T^{1/(1-m)} \tag{3.3.13}$$

where C^* *is given by* (3.3.12)*. Then the Cauchy problem* (3.3.9) *is solvable.*

This result basically settles the issue of existence–non-existence in the radially symmetric case, except for the possible gap between the liminf and the limsup in conditions (3.3.11) and (3.3.13). The analogy with conditions (2.3.2) and (2.3.3) of

Chapter 2, which hold in the slow diffusion case $m > 1$, is obvious. The latter ones are better in the sense that both of them carry lim sup, but they are not as precise, since the constants C and c may differ and they are not optimal as C^* is.

Since the results of existence and non-existence for $m > 1$ do not require radial symmetry, one may think that the condition in Theorem 3.3.2 is just technical. However, this is not the case, as shown via an example in [53]. It is shown that if u_0 vanishes on a "sufficiently wide" logarithmic spiral region, then no local solution exists. In such case the value of the limit in (3.3.11) can be made arbitrarily large. Actually this example can be simplified when $m \leq -1/3$, to yield that if u_0 vanishes on the sector $0 < \theta < l^*$ where $l^* = (m-1)\pi/2m > 0$, then no local solution exists, see Theorem 3.3.4 below. These facts show that the super-diffusive case may hide a rich and possibly very complex non-radial structure behind the existence question, not present in the slow or fast diffusion cases.

Of course a natural direction to investigate is that of finding sufficient conditions for existence in cases where radial symmetry is violated. The following condition for existence has been found in [53].

For a number $\rho > 0$ we denote by G_ρ the Green's function of the ball B_ρ. For a locally bounded function h, we set

$$G_\rho^*(h)(x) = \int_{B_\rho} [\, G_\rho(0, y) - G_\rho(x, y)\,] h(y)\, dy, \quad x \in \bar{B}_\rho.$$

Theorem 3.3.3. *Let $E^* = -2mC^*/(1-m)$. Assume that there exists a non-negative, locally bounded function f for which $u_0 \geq f$ and a sequence $\rho_n \uparrow +\infty$ such that*

$$|x|^{2m/(1-m)} G_{\rho_n}^* f(x) \geq E^* T^{1/(1-m)} + \theta(x), \tag{3.3.14}$$

for all $|x| < \rho_n$. Here $\theta(x)$ is a function such that $\theta(x)|x|^{-2m/(1-m)}$ is locally bounded and $\theta(x) \to 0$ as $|x| \to \infty$. Then problem (3.3.9) is solvable.

This result roughly asserts that if

$$\int (\Gamma|x-y| - \Gamma|y|)\, u_0(y)\, dy \geq E^* T^{1/1-m} |x|^{-2m/1-m}$$

where Γ denotes the fundamental solution of the Laplacian, and the integral is understood in a certain principal value sense, then (3.3.9) is solvable. It is not hard to check that this result implies Theorem 3.3.2 in case that u_0 is radially symmetric. However, it does not answer a basic question: If u_0 is, say, the characteristic function of the sector of angle wider than $2\pi - l^*$, is problem (3.3.9) solvable? It was shown in [55] that the answer to this question is indeed affirmative.

Assume that $n = 2$ and $m < 0$. We consider, for $0 < l \leq 2\pi$, the sector

$$C_l = \{(r\cos\theta, r\sin\theta) : r > 0,\ 0 < \theta < l\} \tag{3.3.15}$$

and denote by χ_l its characteristic function. We define the *critical lenght* (or aperture) l_* as

$$l^* = \min\left\{\frac{(m-1)}{2m}\pi, 2\pi\right\} > 0.$$

Note that $l^* = 2\pi$ iff $m \geq -1/3$. The following result holds.

Theorem 3.3.4. (i) *If* $f \equiv 0$ *a.e. on* C_{l^*}*, then* (3.3.9) *has no local solution.*
(ii) *Let* $l < l^*$ *and* $f = \chi_{2\pi - l}$*. Then* (3.3.9) *admits a solution for* $T = +\infty$*.*

Assertion (ii) of the previous theorem actually follows from the following more general statement, proved in [55].

Let $L^\infty_{2\pi}$ denote the space of 2π-periodic bounded functions on the real line. Let us define, for $0 < l \leq 2\pi$, and $g \in L^\infty_{2\pi}$ the number $K_l(g)$ as

$$K_l(g) = \inf \left\{ \int_I g(\theta)d\theta : I \text{ interval}, \ |I| = l \right\}. \tag{3.3.16}$$

Theorem 3.3.5. *Assume that* $n = 2$*,* $m < 0$*, and that* $f \in L^1_{\text{loc}}(\mathbb{R}^2)$ *satisfies, for some* $R > 0$*, that*

$$u_0(x) \geq r^{-2/(1-m)}\, g(\theta) \tag{3.3.17}$$

for $|x| > R$*, where* $g \in L_{2\pi}$ *is non-negative. Assume also that for some* $l < l^*$ *one has* $K_l(g) > 0$*. Then* (3.3.9) *has a local solution defined at least up to time* T *given by*

$$T = \left(\frac{K_l(g)\,(l^* - l)^4}{C^*} \right)^{1-m}$$

where C^* *is a positive constant dependent only on* m*.*

Note that combining the above theorems we conclude that if $u_0(x) = r^{\frac{-2}{1-m}} g(\theta)$, then (3.3.9) admits a local solution if and only if $K_{l^*}(g) > 0$.

The basic approach in proving this result is based on the connection between the local solvability of the parabolic problem (3.3.9) and the existence of solutions for the elliptic problem

$$\Delta v + v^{-\nu} = u_0(x) \quad \text{in } \mathbb{R}^2, \tag{3.3.18}$$

where $\nu = -1/m > 0$. The relationship between the two problems comes at the formal level from discretization in time of problem (3.3.9) which amounts to solving elliptic problems of the form (3.3.18), see [124]. This was first observed by Vázquez in [124], who proved non-existence for integrable u_0. Existence and non-existence results for (3.3.18), which are indeed analogous to Theorems 3.3.1 and 3.3.2, are contained in [52].

A key fact which allows us to connect the elliptic and parabolic problems is that if u is a positive solution of (3.3.9) then the function $\Phi(x) = |m|^{-1} \int_0^T u^m(x,s)\,ds$ satisfies

$$\Delta\Phi + (\gamma T)^{-1/m}\, \Phi^{1/m} \geq u_0 \tag{3.3.19}$$

with $\gamma = (1-m)/|m|$. Therefore the solvability of (3.3.18) provides through (3.3.19) a control from below of the positive approximations of the maximal solution of (3.3.9).

For the purpose of proving existence of solutions for the elliptic problem (3.3.18) we study first the special case of an u_0 of the form

$$u_0(x) = r^{-2\nu/(1+\nu)}\, g(\theta),$$

with $g \in L^\infty_{2\pi}(\mathbb{R})$ non-negative. In this case we can try a solution of the form

$$v(x) = r^{2/(1+\nu)}\, w(\theta),$$

with w a function in $L^\infty_{2\pi}$. A direct computation shows that w must solve the one dimensional equation

$$w_{\theta\theta} + \beta^2 w + w^{-\nu} = g(\theta) \tag{3.3.20}$$

where $\beta = \pi/l^* = 2/(\nu+1)$, since $\nu = -1/m$. This reduces the problem to solving (3.3.20). The main result for existence of (3.3.20) asserts essentially that if $K_l(g)$ is sufficiently large, for some $l < l^*$, then (3.3.20) is solvable. Its proof relies on a somewhat technical construction of a supersolution for this equation. Having shown the solvability of (3.3.20) one extends it to general elliptic problem and show the analogue of Theorem 3.3.5 for it. This is done through a comparison argument. Finally for the proof of the parabolic result one first solves the problem (3.3.9) with initial data $u_0^\varepsilon = u_0 + \varepsilon$ and then combines the elliptic existence result together with (3.3.19) to show that actually the sequence of solutions $\{u_\varepsilon\}$ converges as $\varepsilon \to 0$ to a strictly positive function which is the desired solution. We refer the reader to the papers [124], [52], [53] and [55] for detailed proofs of all the results mentioned above.

3. Solutions to fast diffusion with "very hot spots". In [36] Chasseigne and Vazquez construct solutions of the Cauchy problem for the fast diffusion equation $u_t = \Delta u^m$ in the super-critical range of exponents $m > m_c = \max\{(n-2)/n, 0\}$ with initial data an arbitrary non-negative Borel measure ν, which is not necessarily a Radon measure. The solutions they construct are continuous as functions into the extended reals and their strongly singular set S, called also "very hot spots", defined as $S = \{x \in \mathbb{R}^n : \nu(B_r(x)) = +\infty\}$ is constant in time. These solutions are constructed as limits of classical solutions.

The existence of non-trivial solutions with "very hot spots" can be extended to the sub-critical case $0 < m \le m_c$, but well-posedness fails [36]. Brezis and Friedman [25] have shown that the limit solution, obtained by approximation by classical solutions, corresponding to a Dirac mass as initial data is constant in time, thus the limit solution vanishes identically for $|x| \ne 0$. More generally, it was shown by Pierre [114] that the limits of classical solutions are not continuous, not even locally integrable functions, when the initial data ν is singular with respect to the $C_{2,\gamma}-$capacity, $\gamma = 1/(1-m)$. Moreover, Pierre constructed, for Radon measures which verify the necessary condition a locally integrable weak solution whose trace is that measure.

4. Extinction in finite time. In [18] Bénilan and Crandall showed that for $0 < m < m_c$, all solutions with data in $L^p(\mathbb{R}^n)$, $p = n\,(1-m)/2$, become zero in finite time. The explicit example

$$u(x,t)=\left(\frac{C\,(T-t)}{|x|^2}\right)^{\frac{1}{1-m}}, \quad C=\frac{2\,m\,[n\,(1-m)-2]}{1-m}$$

has initial data in the Lorentz (Marcinkiewicz) space $L^{p,\infty}(\mathbb{R}^n)$. The extinction property can be extended to data in this space. It does not hold for $m=m_c$.

Chapter 4
The initial Dirichlet problem in an infinite cylinder

This chapter is devoted to the study of the initial Dirichlet problem

$$\left.\begin{aligned} u_t &= \Delta\varphi(u) && \text{in } \Omega = D \times (0,\infty), \\ u &= 0 && \text{on } \partial D \times (0,\infty), \end{aligned}\right\} \tag{4.0.1}$$

where $D \subseteq \mathbb{R}^n$ is an open bounded set with smooth boundary and $\varphi \in \Gamma_a$. We shall study the class of strong solutions of (4.0.1) with $\varphi \in \Gamma_a$.

Definition. We say that u is a strong solution of the IDP (4.0.1) if

(i) $u \in C(\overline{D} \times (0,\infty))$;

(ii) $u \geq 0$;

(iii) $u|_{\partial D \times (0,\infty)} = 0$;

(iv) u satisfies the equation in (4.0.1) in the following sense: for any $\psi \in C^\infty(\overline{\Omega})$ which vanishes on $\partial D \times [\tau_1, \tau_2]$

$$\begin{aligned} &\int_D \int_{\tau_1}^{\tau_2} \left(\varphi(u)\Delta\psi + u\frac{\partial\psi}{\partial t}\right) dx dt \\ &\qquad = \int_D u(x,\tau_2)\,\psi(x,\tau_2) dx - \int_D u(x,\tau_1)\,\psi(x,\tau_1) dx \end{aligned} \tag{4.0.2}$$

for all $0 < \tau_1 < \tau_2 < \infty$.

It is clear that strong solutions are weak solutions as they were introduced in Section 1.1 of Chapter 1, in any cylinder of the form $D \times (\tau,\infty)$ with $\tau > 0$.

4.1 Preliminary results

We begin this section of preliminary results with a compactness lemma which will be used throughout this chapter.

Lemma 4.1.1. *Suppose $\{u_j\}$ is a sequence of strong solutions of the IDP (4.0.1) in $\Omega = D \times (0,\infty)$. Suppose that given $t > 0$ there exists $M_t > 0$ such that*

$$\sup_{x \in D} u_j(x,t) \leq M_t.$$

Then there exists a subsequence $\{j_k\}$ and a strong solution u of the IDP (4.0.1) in Ω such that $\{u_{j_k}\}$ converges uniformly to u in $\overline{D} \times [s,\tau]$, for all $0 < s < \tau < +\infty$.

Proof. From Theorem 1.1.1 of Chapter 1 it follows that

$$\sup_{D\times[\frac{s}{2},2\tau]} u_j(x,t) \le M_{\frac{s}{2}}.$$

Using the equicontinuity result, Theorem 1.5.1 of Chapter 1, we have that there exists $\{u_{j_k}\}$ such that u_{j_k} converges to u uniformly on $K\times[s,\tau]$ for every $K \subset\subset D$. To extend the equicontinuity result to $\bar{D}\times[s,\tau]$ one needs to observe that a slight modification of the proof of Theorem 1.5.1 gives a modulus of continuity up to the lateral part of the boundary when the lateral values are zero. □

The following result follows from the proof of the uniqueness theorem in [113].

Lemma 4.1.2 ([113]). *Let u_1, u_2 be strong solutions of the IDP of (4.0.1) in $D\times(0,\infty)$. Let η be a $C_0^\infty(\Omega)$ positive function, and $\{\tau_j\}$ be a sequence with $\tau_0 = T < \infty$ such that $\tau_j \to 0$ as $j\to\infty$. Then there is a sequence of measures $\{\lambda_j\}$ in D such that*

$$\int_D d\lambda_j \le \int_D \eta\, dx$$

and

$$\int_D [w_1(x,T) - w_2(x,T)]\,\eta(x)\,dx = \int_D [w_1(x,\tau_j) - w_1(x,\tau_j)]\,d\lambda_j \qquad (4.1.1)$$

where $w_i = Gu_i$ denotes the Green's potential of u_j on D.

Proof. Similarly to the proof of Theorem 1.1.1 in Chapter 1, we define

$$A = \begin{cases} \frac{\varphi(u_1)-\varphi(u_2)}{u_1-u_2} & u_1 \ne u_2,\\ \varphi'(u_1) & u_1 = u_2. \end{cases}$$

Thus $A \ge 0$. Let $\{\varepsilon_k\}$ be a sequence of real numbers such that $\varepsilon_k > 0$ and $\varepsilon_k \to 0$. Set

$$\alpha_k = \frac{|\varphi(u_1)-\varphi(u_2)|}{|u_1-u_2|+\varepsilon_k}$$

and

$$A_k = \alpha_k + \varepsilon_k$$

so that A_k is continuous and strictly positive in $D\times(0,\infty)$.

Let θ be the Green's potential of η, i.e.

$$\begin{aligned} \Delta\theta &= -\eta \quad &&\text{in } D,\\ \theta &= 0 &&\text{at } \partial D. \end{aligned}$$

For a smooth positive function α in $\mathbb{R}^n\times[0,\infty)$, define $S(\alpha) = \psi$ as the solution of the linear (backward parabolic problem)

$$\left.\begin{aligned} \psi_t + \alpha\,\Delta\psi &= 0 \qquad \text{in } D\times(0,T),\\ \psi(x,T) &= \theta(x) x \in D,\\ \psi &= 0 \qquad \text{on } \partial D\times(0,T). \end{aligned}\right\} \qquad (4.1.2)$$

Then $h = \Delta\psi$ is a solution of the problem

$$\left.\begin{aligned} h_t + \Delta(\alpha\, h) = 0 \quad & \text{in } D \times (0, T), \\ h(x, T) = -\eta(x) \quad & x \in D, \\ h = 0 \quad & \text{on } \partial D \times (0, T). \end{aligned}\right\} \tag{4.1.3}$$

Moreover

$$\begin{aligned} \int_D \int_0^T \alpha h^2 &= \iint \alpha (\Delta\psi)^2 \\ &= -\iint \frac{\partial\psi}{\partial t}\, \Delta\psi \\ &= -\int_D \theta \Delta\theta\, dx + \int_D \psi(x, 0) \Delta\psi(x, 0)\, dx + \iint \psi\, \frac{\partial(\Delta\psi)}{\partial t}. \end{aligned}$$

Since

$$-\int \theta\, \Delta\theta\, dx = \int |\nabla\theta|^2\, dx$$

and

$$\int \psi(x, 0)\, \Delta\psi(x, 0)\, dx \le 0$$

and

$$\frac{\partial(\Delta\psi)}{\partial t} = \frac{\partial h}{\partial t} = -\Delta(\alpha\, h)$$

we conclude that

$$\int_D \int_0^T \alpha h^2 \le \int_D |\nabla\theta|^2 - \int_D \int_0^T \alpha h^2.$$

Hence

$$\int_D \int_0^T \alpha\, h^2\, dt dx \le \frac{1}{2} \int_D |\nabla\theta|^2\, dx. \tag{4.1.4}$$

Now we want to solve the parabolic backward problem (4.1.3) with $\alpha = A$.

Define $b = u_1 - u_2$, so that

$$\begin{aligned} \int_D b(x, T)\theta(x)\, dx - \int_D b(x, \tau)\, \psi(x, \tau)\, dx \\ &= \int_D [w_1(x, T) - w_2(x, T)]\; \eta(x)\, dx \\ &\quad - \int_D [w_1(x, \tau) - w_2(x, \tau)]\; h(x, \tau)\, dx \\ &= \int_D \int_0^T \left([\varphi(u_1) - \varphi(u_2)]\; h + b\, \frac{\partial\psi}{\partial t} \right) dt dx \\ &= \int_D \int_0^T b\, (A - \alpha)\, \Delta\psi\, dt dx. \end{aligned} \tag{4.1.5}$$

Notice that

$$\begin{aligned}\frac{d}{dt}\int_D |h(x,t)|\,dx &= -\int_D h_t\,dx\\ &= \int_D \Delta(\alpha h)dx\\ &= \int_{\partial D}\frac{\partial}{\partial n}(\alpha h)\,d\sigma \ge 0\end{aligned}$$

since $h(x,t) \le 0$ in D and vanishes on $\partial D \times (0,T)$. Therefore

$$\int_D |h(x,t)|\,dx \le \int_D \eta(x)dx. \tag{4.1.6}$$

Let $\{\alpha_\nu\} \subset C^\infty(\mathbb{R}^n \times [0,\infty))$, with $\alpha_\nu > 0$ such that $\alpha_\nu \to A_k$ uniformly on compact subsets of $\overline{D} \times (0,\infty)$. From (4.1.6), for each k, j there exists a measure $\lambda_j^{(k)}$ defined as the weak limit of $\{-h_{\nu'}(x,\tau_j)dx\}_{\nu'=0}^\infty$ for some subsequence $\{\nu'\}$. Moreover,

$$\int_D d\lambda_j^{(k)} \le \int_D \eta(x)\,dx.$$

Using (4.1.5) as in the proof of Theorem 1.1.1 in Section 1.1 of Chapter 1 we obtain that

$$\begin{aligned}&\left|\int_D (w_1(x,T)-w_2(x,T))\,\eta(x)\,dx - \int_D (w_1(x,\tau)-w_2(x,\tau))\,d\lambda_j^k\right|\\ &\qquad \le C(\eta)\left(\int_\Omega\int_{\tau_j}^T \frac{(A-A_k)^2}{A_k}\,b^2\,dxdt\right)^{1/2}.\end{aligned}$$

Since $(A-A_k)^2b^2/A_k \le C\,\varepsilon_k$, letting $k\to\infty$ we obtain the desired result. □

Remark. If u is a strong solution of the IDP in Ω and w is the Green's potential of u then

$$\frac{\partial w}{\partial t} = -\varphi(u) \le 0$$

in the distribution sense, and

$$e(x,\tau,T) \equiv w(x,T) - w(x,\tau) + \int_\tau^T \varphi\,(u(x,t))\,dt$$

is a well defined continuous function with

$$e(x,\tau,T) = 0 \quad \text{for } x \in \partial D.$$

Moreover

$$\Delta_x e(x,\tau,T) = 0$$

in the distribution and hence

$$e(x,\tau,T)\equiv 0$$

which shows that

$$w(x,T)=w(x,\tau)-\int_\tau^T \varphi\,(u(x,t))\,dt.$$

Thus $\lim_{t\downarrow 0} w(x,t)$ exists (it may be infinite) for each $x\in D$.

Lemma 4.1.3 (Maximum principle [113]). *Let u_1, u_2 be strong solutions of the IDP (4.0.1) in $\Omega = D\times(0,\infty)$. Suppose that $u_2\in C\big(\bar{D}\times[0,\infty)\big)$ and that the Green's function $w_i = Gu_i$ verify that*

$$\lim_{t\downarrow 0} w_2(x,t)\le \lim_{t\downarrow 0} w_1(x,t).$$

Then $w_2(x,t)\le w_1(x,t)$ for $(x,t)\in\Omega$.

Remark. Notice that by hypothesis $\lim_{t\downarrow 0} w_2(x,t)$ is finite for any $x\in D$.

Proof. Fix $T>0$, and let $\{\tau_j\}$ be a decreasing sequence such that $\tau_0 = T$ and $\tau_j\to 0$. Set $\eta\in C_0^\infty(D)$, $\eta\ge 0$. Let $\{\lambda_j\}$ be the sequence of measures provided by Lemma 4.1.2. Passing to a subsequence we find that there exists λ_∞ such that

$$\int_D d\lambda_\infty \le \int_D \eta\,dx$$

and

$$\lim_{j\to\infty}\int_D \theta\,d\lambda_j = \int_D \theta\,d\lambda_\infty$$

for any $\theta\in C(\bar{D})$ with $\theta|_{\partial D}=0$. Our goal is to show that

$$E=\int_D \big[w_1(x,T)-w_2(x,T)\big]\,\eta(x)\,dx\ge 0.$$

From (4.1.1)

$$E=\int_D \big[w_1(x,\tau_j)-w_2(x,\tau_j)\big]\,d\lambda_j.$$

Taking $k\le j$ and using that $\partial w_1/\partial t\le 0$ it follows that

$$E\ge \int_D w_1(x,\tau_k)\,d\lambda_j - \int_D w_2(x,\tau_j)\,d\lambda_j.$$

Notice that from our hypothesis

$$\lim_{j\to\infty}\int_D w_2(x,\tau_j)\,d\lambda_j = \int_D h_2(x)\,d\lambda_\infty$$

where $h_2(x) = \lim_{t\downarrow 0} w_2(x,t)$. Hence

$$E \geq \int_D w_1(x,\tau_k)\,d\lambda_\infty - \int_D h_2(x)\,d\lambda_\infty.$$

Combining the Monotone Convergence Theorem and our hypothesis we obtain that

$$E \geq \int_D [h_1(x) - h_2(x)]\,d\lambda_\infty \geq 0$$

where $h_1(x) = \lim_{t\downarrow 0} w_1(x,t)$, which completes the proof. □

4.2 The friendly giant (slow diffusion)

In this section we shall prove the existence of a strong solution $\alpha(x,t)$ in the slow diffusion case $\varphi \in \mathcal{S}_a$ (as defined in the Introduction) without an initial trace at $t = 0$ [50]. For a function $f \in C(\bar{D})$ with $f|_{\partial D} = 0$ and $f \geq 0$, let us denote by u_f the unique solution of the IVP

$$\left.\begin{aligned} u_t &= \Delta\varphi(u) && \text{in } D \times (0,\infty),\\ u &= 0 && \text{on } \partial D \times (0,\infty),\\ u(x,0) &= f(x) && x \in D, \end{aligned}\right\} \tag{4.2.1}$$

in $C(\bar{D} \times [0,\infty))$. The existence of u_f follows by an argument similar to that given in Section 1.6 of Chapter 1 and the uniqueness is a consequence of Theorem 1.1.1 of Chapter 1.

Theorem 4.2.1. *Let $\varphi \in \mathcal{S}_a$. Suppose $f \in C(\bar{D})$ with $f|_{\partial D} = 0$, and $f \geq 0$. Let u_f be the unique solution of the IVP (4.2.1) in $C(\bar{D} \times [0,\infty))$. Define*

$$\alpha(x,t) \equiv \sup_f u_f(x,t).$$

Then $\alpha(x,t)$ is a strong solution of the IDP (4.0.1) in $D \times (0,\infty)$. Moreover, if u is a strong solution of the IDP (4.0.1) in $D \times (0,\infty)$ then

$$u(x,t) \leq \alpha(x,t).$$

Proof. Denote by β the inverse of φ ($\varphi \in \mathcal{S}_a$) and by ψ its primitive, i.e.

$$\psi(z) = \int_0^z \beta(s)\,ds.$$

Notice that $\varphi(s) \geq s^{1+a}$ for $s \geq 1$, since

$$\begin{aligned} \log\varphi(s) &= \int_1^s (\log\varphi(r))'\,dr = \int_1^s \frac{\varphi'(r)}{\varphi(r)}\,dr \\ &= \int_1^s \frac{r\,\varphi'(r)}{\varphi(r)}\,\frac{dr}{r} \geq (1+a)\,\log s. \end{aligned}$$

We claim that

$$\psi(z) \le C\,z^{1+\gamma} \quad \text{for } z \ge 1, \text{ with } \gamma \in (0,1). \tag{4.2.2}$$

Since $\varphi(s) \ge s^{1+a}$, it follows that

$$s \ge \beta\big(s^{1+a}\big). \tag{4.2.3}$$

Defining $r = s^{1+a}$ we have that

$$r^{1/(1+a)} \ge \beta(r) \quad \text{for } r \ge 1$$

and

$$\begin{aligned}\psi(z) &= \int_0^1 \beta(s)\,ds + \int_1^z \beta(s)\,ds \\ &\le \int_0^1 \beta(s)ds + \int_1^z s^{1/(1+a)}ds \\ &\le C\,z^{1+\gamma}\end{aligned}$$

for $z \ge 1$, which proves the claim (4.2.2).

Let A be a bounded domain such that $D \subset\subset A$. Denote by $v = v_A$ the solution of the variational problem (nonlinear eigenvalue problem)

$$\sup\Big\{ \int_A \psi(w)\,dx : w \in H_0^1(A) \text{ with } \int_A |\nabla w|^2\,dx = 1 \Big\}.$$

Since $\psi(z) = \int_0^z \beta(s)ds \le C\,z^2 + C'$ and $H_0^1(A) \hookrightarrow L^2(A)$ is compact, the supremum is attained in v, and v solves the variational Euler equation

$$\Delta v = -\mu\,\beta(v)$$

where μ is the Lagrange multiplier. Thus the following hold.

(i) $v = v_A > 0$ in A. Otherwise $|v|$ would increase $\int_A \psi(v)\,dx$.

(ii) $\mu > 0$. This follows by integrating the equation $\Delta v = -\mu\,\beta(v)$ multiplied by v.

(iii) $v \in C^\infty(\bar{A})$ by classical regularity theorems.

Define $u(x,t)$ by the formula

$$\varphi\,(u(x,t)) = \frac{v(x)}{t^\gamma} \tag{4.2.4}$$

for a constant $\gamma > 0$ to be chosen in the sequel and also let

$$m = \inf_D v(x) > 0.$$

Claim. There exists $T_0 > 0$ and $\gamma_0 > 0$ such that u in (4.2.4) is a supersolution of $\partial z/\partial t = \Delta\varphi(z)$ in $D \times (0, T_0)$ (i.e., $\Delta\varphi(u) - \partial u/\partial t \le 0$) when $\gamma \ge \gamma_0$.

To prove this claim, first we choose T_1 so small that if $t \in (0, T_1)$ then

$$\frac{v(x)}{t^\gamma} \geq 1 \quad \text{for } (x,t) \in D \times (0, T_1) \text{ and } \gamma \geq \gamma_0.$$

Next we compute that

$$\Delta\varphi(u) - \frac{\partial u}{\partial t} = -\mu \frac{\beta(v)}{t^\gamma} + \gamma\, \beta'\left(\frac{v}{t^\gamma}\right) \frac{v}{t^{\gamma+1}}.$$

Since $\varphi \in \mathcal{S}_a$ it follows that

$$\beta'(s) \leq C \frac{\beta(s)}{s} \qquad \text{for } s > 0$$

for some $C > 0$. Hence

$$\Delta\varphi(u) - \frac{\partial u}{\partial t} \leq -\mu \frac{\beta(v)}{t^\gamma} + C \frac{\gamma v}{t^{1+\gamma}} \frac{\beta\left(v/t^\gamma\right)}{v/t^\gamma}$$

since $v/t^\gamma \geq 1$. Thus (4.2.3) yields

$$\begin{aligned} \Delta\varphi(u) - \frac{\partial u}{\partial t} &\leq -\mu \frac{\beta(v)}{t^\gamma} + C \frac{\gamma v}{t^{1+\gamma}} \frac{v^{\frac{-a}{1+a}}}{t^{\frac{-a\gamma}{1+a}}} \\ &\leq -\frac{\mu C'}{t^\gamma} + \frac{C''}{t^{1+\frac{\gamma}{1-a}}}. \end{aligned}$$

Choosing γ sufficiently large we complete the proof of the claim.

Now by Theorem 1.1.1 of Chapter 1 (which also works for supersolutions) and (4.2.3) we get

$$u_f(x,t) \leq u(x,t) \leq \frac{C}{t^\gamma} \quad \text{for all } t \in (0, T_0)$$

where C depends only on φ. Also for $t \geq T_0$ from the maximum principle we have

$$u_f(x,t) \leq C_1.$$

Pick $K_n \subset\subset K_{n+1} \subset\subset D$ such that $\bigcup_1^\infty K_n = D$. Choose $\{f_n\}_1^\infty$ an increasing sequence in $C_0^\infty(D)$ such that $f_n \equiv n$ on K_n and support $f_n \subseteq K_{n+1}$. From Theorem 1.1.1 it follows that there exists a subsequence $\{u_{f_{n_j}}\}$ which converges uniformly to $\overline{\alpha}(x,t)$ on compact sets of $D \times (0, \infty)$ and where $\overline{\alpha}(x,t)$ is a strong solution of the IDP (4.0.1).

We shall show that

$$\overline{\overline{\alpha}}(x,t) := \sup_{f \in C_0(D)} u_f = \overline{\alpha}(x,t). \tag{4.2.5}$$

Indeed, by definition $\overline{\overline{\alpha}} \geq \overline{\alpha}$. On the other hand, given $f \in C_0(D)$, there exists n_j such that $f_{n_j} \geq f$. Thus by Theorem 1.1.1 of Chapter 1, $u_f \leq \overline{\alpha}$, and (4.2.5) is proved.

Finally, given $f \in C(D)$ with $f|_{\partial D} = 0$ there exists $\{f_j\} \subseteq C_0(D)$ such that $\{f_j\}$ is a non-decreasing sequence of functions which converges uniformly on D. Thus,

$$\alpha(x,t) = \overline{\overline{\alpha}}(x,t) = \sup_f u_f(x,t).$$

Now let u be a strong solution of the IDP (4.0.1). Then

$$u_n(x,t) := u(x, t + 1/n) \leq \alpha(x,t).$$

Letting n tend to infinity it follows that

$$u(x,t) \leq \alpha(x,t)$$

and the proof of the theorem is finished. □

The next results characterize the exceptional solution $\alpha(x,t)$.

Lemma 4.2.2. *Let $u(x,t)$ be a strong solution of the IDP (4.0.1) in $D \times (0,\infty)$ such that $w(x,t) = Gu(x,t)$ (Green's potential of u) satisfies*

$$\lim_{t \downarrow 0} w(x,t) = +\infty \quad \textit{for all } x \in D.$$

Then $u \equiv \alpha$.

Proof. By the previous lemma $u \leq \alpha$. Let $\{u_n\}$ be a sequence of strong solutions such that

(i) $u_n \in C([0,\infty) \times \overline{D})$,

(ii) $0 \leq u_n \leq u_{n+1}$,

(iii) $\lim_{n\to\infty} u_n(x,t) = \alpha(x,t)$ for $(x,t) \in D \times (0,\infty)$.

This sequence was constructed in the proof of Theorem 4.2.1. We apply the maximum principle (Lemma 4.1.3) to w and w_n (which is continuous up to $t = 0$). Since

$$\lim_{t \downarrow 0} w(x,t) \geq \lim_{t \downarrow 0} w_n(x,t)$$

it follows that $w(x,t) \geq w_n(x,t)$. Therefore $w(x,t) \geq G\alpha(x,t)$. □

Lemma 4.2.3. *Let $u(x,t)$ be a strong solution of the IDP (4.0.1) in $D \times (0,\infty)$. Then $u \not\equiv \alpha$ if and only if*

$$\sup_{t>0} \int_D u(x,t)\delta(x)\,dx < \infty \tag{4.2.6}$$

where $\delta(x) = \mathrm{dist}(x, \partial D)$.

Proof. Assume $u \not\equiv \alpha$. From the previous lemma there exists $x_0 \in D$ such that

$$\lim_{t \downarrow 0} w(x_0, t) < \infty \quad (\text{where } w(x,t) = Gu(x,t)).$$

Since $w(x, \cdot)$ is a non-increasing function we have that

$$\sup_{t>0} w(x_0, t) = \sup_{t>0} \int_D G(x_0, y) u(y, t) dy < \infty.$$

On the other hand as a consequence of the Hopf's maximum principle (see [76])

$$G(x_0, y) \geq C\,\delta(y).$$

Therefore

$$\sup_{t>0} \int_D u(y,t)\,\delta(y)\,dy < \infty.$$

Now we assume that the inequality (4.2.6) holds. Define

$$a(x) = \int_D G(x,y)\,dy$$

so that

$$\begin{aligned} \Delta a &= -1 \quad \text{in } D, \\ a &= 0 \qquad \text{on } \partial D, \end{aligned}$$

and $a(x) \simeq \delta(x)$. In fact: for $y \in D$ such that $|x-y| \geq \delta(x)/2$

$$G(x,y) \cong \frac{\delta(x)\delta(y)}{|x-y|},$$

and for $y \in D$ such that $|x-y| < \delta(x)/2$

$$G(x,y) \cong \frac{1}{|x-y|^{n-2}}.$$

The proof of these results follow by comparison with the Green's functions in an exterior ball.

Fubini's theorem and the symmetry of the Green's function lead to

$$\int_D w(x,t) dx \equiv \int_D u(x,t)\,a(x)\,dx.$$

Now taking the supremum and using (4.2.6) we obtain that

$$\sup_{t>0} \int_D w(x,t) < \infty \tag{4.2.7}$$

which implies that $u \not\equiv \alpha$. □

Remark. We have also proven that if $u \not\equiv \alpha$ then (4.2.7) holds.

4.3 The trace (slow diffusion)

By removing the exceptional solution $\alpha(x,t)$ constructed in the previous section from the class of strong solutions in $D \times (0,\infty)$ we shall now prove the existence of the initial trace for strong solutions of IDP (4.0.1) with $\varphi \in \mathcal{S}_a$.

Theorem 4.3.1. *Let $u(x,t)$ be a strong solution of the IDP (4.0.1), with $\varphi \in \mathcal{S}_a$. If $u \not\equiv \alpha$ (with α as in Theorem 4.2.1) then there exist non-negative Borel measures μ and λ on D and ∂D respectively with*

$$\int_D \delta(x)\, d\mu(x) < \infty \quad \text{and} \quad \int_{\partial D} d\lambda < \infty$$

such that for any $\eta \in C_0^\infty(\mathbb{R}^n)$ with $\eta|_{\partial D} = 0$

$$\lim_{t\downarrow 0} \int_D u(x,t)\eta(x)dx = \int_D \eta(x)\, d\mu(x) - \int_{\partial D} \frac{\partial \eta}{\partial n}(Q)\, d\lambda(Q)$$

where $\partial/\partial n$ denotes the exterior normal derivative to ∂D.

Proof. By the monotonicity of $w(x,\cdot) = Gu(x,\cdot)$ for any $x \in D$ and the above remark it follows that

$$\lim_{t\downarrow 0} w(x,t) \equiv h(x)$$

exists for all $x \in D$ with $h \geq 0$, superharmonic, and by the Monotone Convergence Theorem

$$\|w(x,t) - h(x)\|_{L^1(D)} \to 0 \quad \text{as } t \downarrow 0.$$

In particular, $h \in L^1(D)$. Also

$$\int_D u(x,t)\,\eta(x)\,dx = -\int_D w(x,t)\Delta\,\eta\,dx \to -\int_D h(x)\Delta\eta(x)dx$$

when $t \downarrow 0$. From Classical Potential Theory we have the Riesz decomposition of h (see [82])

$$h(x) = G\mu(x) + \alpha(x)$$

where α is harmonic and non-negative and μ is a measure such that

$$\int_D \delta(x)\, d\mu(x) < \infty. \tag{4.3.1}$$

To obtain λ we use the Martin's representation theorem of harmonic positive functions (see [82]). In fact, in our case due to the regularity of D the Martin boundary agrees with the Euclidean boundary. Thus

$$\alpha(x) = \int_{\partial D} K(x,Q)\, d\lambda(Q).$$

where $K(\cdot,\cdot)$ is the Martin's kernel for D Now

$$\int_D \left(\int_D G(x,y)\,d\mu(y)\right)\Delta\eta(x)dx = -\int \eta(y)\,d\mu(y) \tag{4.3.2}$$

and

$$\int_D \left(\int_{\partial D} K(x,Q)d\lambda(Q)\right)\Delta\eta(x)dx = \int_{\partial D} \frac{\partial\eta(Q)}{\partial n}\,d\lambda(Q). \tag{4.3.3}$$

To justify these two inequalities we proceed as follows: For the first one, note that (4.3.1) implies that

$$\int_D \left|\int_D G(x,y)d\mu(y)\right| dx < \infty.$$

Hence it suffices to verify (4.3.2) in the case when $d\mu(y) = f(y)\,dy$, with $f \in C_0^\infty(D)$. But in this case the result follows from the Green's formula. For the second one (4.3.3) using a well-known estimate for the kernel $K(x,Q)$ (see for example [137]) it follows that

$$\int_D \int_{\partial D} K(x,Q)d\lambda(Q)dx < \infty.$$

Hence it suffices to verify (4.3.3) for $d\lambda(Q) = f(Q)\,dw(Q)$, where $f \in C^\infty(\partial D)$ and $dw(Q)$ is the harmonic measure for D. In this case the identity follows again by Green's formula. □

The following result shows that strong solutions are uniquely determined by their initial trace.

Theorem 4.3.2. *Suppose u_1, u_2 are two strong solutions of the IDP* (4.0.1) *such that*

$$\lim_{t\downarrow 0} \int_D u_1(x,t)\,\eta(x)\,dx = \lim_{t\downarrow 0} \int_D u_2(x,t)\,\eta(x)\,dx$$

for any $\eta \in C_0^\infty(\mathbb{R}^n)$, $\eta \equiv 0$ on ∂D. Then $u_1 \equiv u_2$.

Proof. Assume first that there exists $\eta \in C_0^\infty(\mathbb{R}^n)$, $\eta \equiv 0$ on ∂D such that

$$\lim_{t\downarrow 0} \int_D u_i(x,t)\,\eta(x)\,dx = \infty.$$

Then by Theorem 4.3.1 we have $u_1 \equiv u_2 \equiv \alpha$ the exceptional solution constructed in the previous section. Thus we can assume that

$$\lim_{t\downarrow 0} \int_D u_i(x,t)\,\eta(x)\,dx < \infty$$

for $i = 1, 2$ and any η.

Let $\theta \in C_0^\infty(D)$, $\theta \geq 0$, and define

$$h_i(x) = \lim_{t\downarrow 0} w_i(x,t).$$

By the Monotone Convergence Theorem

$$\int_D h_i(x)\,\theta(x)\,dx = \lim_{t\downarrow 0}\int_D w_i(x,t)\,\theta(x)\,dx$$
$$= \lim_{t\downarrow 0}\int_D u_i(x,t)\,\eta(x)\,dx$$

where $\eta(x) = G\theta(x)$. Thus $h_1 = h_2$ a.e. But they are potentials, therefore every point is a Lebesgue point and $h_1 \equiv h_2$.

Now for $s > 0$ define

$$w(x,t) = w_2(x, t+s)$$

continuous up to $t = 0$ with

$$\lim_{t\downarrow 0} w_1(x,t) = \lim_{t\downarrow 0} w_2(x,t) \geq w_2(x,s) \equiv \lim_{t\downarrow 0} w(x,t).$$

Consequently by the maximum principle (Lemma 4.1.3)

$$w_1(x,t) \geq w(x,t) = w_2(x,t+s) \quad \text{for all } s > 0.$$

Hence $w_1(x,t) \geq w_2(x,t)$. By symmetry of the argument, $w_1 \equiv w_2$. □

4.4 Existence of solutions (slow diffusion)

This section is devoted to establishing the existence of solutions to IDP (4.0.1) with $\varphi \in \mathcal{S}_a$ as stated next.

Theorem 4.4.1. *Let $\varphi \in \mathcal{S}_a$. Given a pair of measures μ on D and λ on ∂D with $\mu \geq 0$, $\lambda \geq 0$, $\int_D \delta(x)\,d\mu(x) < \infty$ and $\int_{\partial D} d\lambda < \infty$, there exists a unique strong solution u of the IDP (4.0.1) in $D \times (0,\infty)$ such that for any $\eta \in C^\infty(\mathbb{R}^n)$, $\eta \equiv 0$ on ∂D*

$$\lim_{t\downarrow 0}\int_D u(x,t)\,\eta(x)dx = \int_D \eta\,d\mu - \int_{\partial D}\frac{\partial\eta}{\partial n}\,d\lambda$$

where $\partial/\partial n$ denotes the exterior normal derivative to ∂D.

Proof. The proof will be carried out in two steps:

Step 1. Assume $\lambda \equiv 0$ and $\operatorname{supp}\mu \subset K \subset\subset D$. Denote by u_f the solution of the BVP (see Section 1.6 of Chapter 1)

$$\begin{aligned} u_t &= \Delta\varphi(u) && \text{in } D\times(0,\infty),\\ u &= 0 && \text{on } \partial D\times(0,\infty),\\ u(x,0) &= f(x) && x \in D, \end{aligned}$$

where $f \in C_0^\infty(D)$, $f \geq 0$. Thus $u_f \in C(\bar{D}\times[0,\infty))$.

Claims.

(i) $M = \sup_{t>0} \int_D u_f(x,t)dx \le \int_D f(x)$.

(ii) If $\eta \in C^\infty(\mathbb{R}^n)$, $\eta \equiv 0$ on ∂D, then there exists $\alpha > 0$ such that for $t \in (0,1)$

$$\left| \int_D [u_f(x,t) - f(x)]\, \eta(x) dx \right| \le C_{M,\eta}\, t^\alpha.$$

Proof of (i). Formally

$$\begin{aligned} \frac{d}{dt} \int_D u_f(x,t)\, dx &= \int_D \Delta\varphi(u_f(x,t))\, dx \\ &= \int_{\partial D} \frac{\partial}{\partial n} \varphi(u_f(x,t))\, d\sigma(x) \le 0. \end{aligned}$$

The above formal computation can be justified by approximating φ by φ_j with $\varphi_j > 0$ at zero, such that $\varphi_j \to \varphi$ uniformly on compacts, passing to the limit and using Fatou's Lemma. Thus we obtain Claim (i).

Proof of (ii). Define

$$v(x,t) = \begin{cases} u_f(x,t) & x \in D, \\ 0 & x \notin D. \end{cases}$$

Thus v is a subsolution of $\partial u/\partial t = \Delta\varphi(u)$ in $\mathbb{R}^n \times (0,\infty)$. From Claim (i) and Lemma 1.3.1 in Section 1.3 of Chapter 1, we have

$$\varphi(v) \le 1 + C\, v\, t^{-\sigma} \quad \text{for } t \in (0,1) \tag{4.4.1}$$

where $C = C(M,a)$ and $\sigma = \sigma(M,a) \in (0,1)$. Using the identity

$$\int_D [u_f(x,t) - f(x)]\, \eta(x) d\, x = \int_0^t \int_D \varphi(u_f) \Delta\eta\, dx dt$$

and (i) we obtain (ii).

To justify the above proof we observe that in the proof of (4.4.1) (Lemma 1.3.1 of Chapter 1) only the fact that v is a subsolution of $\partial z/\partial t = \Delta\varphi(z)$ was used.

Now pick $\{f_j\} \subseteq C_0^\infty(D)$, $f_j \ge 0$ such that $\int_D f_j dx \le \int_D d\mu$, supp $f_j \subset \overline{K} \subset D$, and $f_j\, dx \to d\mu$ weakly. Let $u_j = u_{f_j}$. Since $u_j \le \alpha$, by Lemma 4.1.1 there exists a subsequence (which we still denote $\{u_j\}$) and a strong solution u of the IDP (4.0.1) such that $\{u_j\}$ converges uniformly to u in $\overline{D} \times [s,\tau]$ for any $0 < s < \tau < \infty$. Thus

$$\begin{aligned} &\int_D u(x,t)\eta(x)\, dx - \int_D \eta(x)\, d\mu \\ &= \int_D [u(x,t) - u_j(x,t)]\, \eta(x)\, dx + \int_D [u_j(x,t) - f_j(x)]\, \eta(x)\, dx \\ &\quad + \int_D \eta(x)[f_j(x)\, dx - d\mu(x)] \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

By the above estimates I + II + III → 0 as $t \downarrow 0$, completing the proof of Step 1.

Step 2. General case. We introduce the superharmonic function

$$h(x) = \int_D G(x, y)\, d\mu(y) + \int_{\partial D} K(x, Q)\, d\lambda(Q)$$

(see the notation in the proof of Theorem 4.3.1). As in the proof of Theorem 4.3.1 it follows that $h \geq 0$ with $\int_D h(x)dx < \infty$.

Now we choose a sequence of domains $\{D_j\}_1^\infty$ such that $D_j \subset\subset D_{j+1}$ and $D = \bigcup_{j=1}^\infty D_j$. Define

$$h_j = \begin{cases} h & \text{in } \bar{D}_j, \\ \text{harmonic} & \text{in } D - \bar{D}_j, \\ 0 \text{ on } \partial D & \text{and } h_j \text{ on } \partial D_j. \end{cases}$$

By construction h_j is superharmonic (satisfies the submean value property), $h_j \uparrow h$ as $j \uparrow \infty$, and

$$h_j = \int_D G(x, y) d\mu_j(y) \quad \text{with supp}\, \mu_j \subseteq \bar{D}_j$$

(since the Laplacian of h_j is a measure that takes the value zero continuously on ∂D).

Let u_j be the solution corresponding to μ_j, constructed in Step 1. Denote by $w_j(x, t)$ the Green's potential of u_j (i.e. $w_j(x, t) = Gu_j(x, t)$). Hence

$$\lim_{t \downarrow 0} w_j(x, t) = h_j(x).$$

Moreover, $u_j(x, t) \leq \alpha(x, t)$ (exceptional solution), and hence by Lemma 4.1.1, there exists a subsequence (which we still denote by $\{u_j\}$) which converges uniformly on $\bar{D} \times [s, \tau]$ to a strong solution u of the IDP (4.0.1).

Thus u is our candidate for solution. It remains to check that it has the right trace, in other words

$$\lim_{t \downarrow 0} w(x, t) = h(x) \tag{4.4.2}$$

Observe first that since $w(x, t) = \lim_{j \to \infty} w_j(x, t)$ (uniform limit) and $w_j(x, t) \leq h_j(x) \leq h(x)$, then for $u \not\equiv \alpha$, $w(x, t) = Gu(x, t) \leq h(x)$ for all $t > 0$. We claim that

$$w(x, t) \geq w_j(x, t). \tag{4.4.3}$$

Indeed, we know that $w(x, t) = \lim_{k \to \infty} w_k(x, t)$. Choose $k > j$, $s > 0$ and apply Lemma 4.1.3 to $w_k(x, t)$ and $w_j(x, t + s)$. Since

$$\lim_{t \downarrow 0} w_k(x, t) = h_k(x) \geq h_j(x) \geq w_j(x, s)$$

it follows that

$$w_k(x, t) \geq w_j(x, t + s) \quad \text{for all } s > 0$$

which proves (4.4.3). Thus

$$h(x) \geq \lim_{t \downarrow 0} w(x,t) \geq \lim_{t \downarrow 0} w_j(x,t) = h_j(x)$$

and therefore

$$h(x) = \lim_{t \downarrow 0} w(x,t).$$

which completes the proof of Theorem 4.3.1. □

Corollary 4.4.2. *Let u be a strong solution of the IDP in $D \times (t_1, t_2)$. Then there exists a strong solution $\tilde{u}$ of the IDP in $D \times (t_1, \infty)$ such that $u \equiv \tilde{u}$ in $D \times (t_1, t_2)$.*

Proof. Without loss of generality we can assume $t_1 = 0$. Let $w(x,t) = Gu(x,t)$. If $\lim_{t \downarrow 0} w(x,t) \equiv \infty$ then $u \equiv \alpha$ and we complete the proof.

Otherwise by the previous result there exists a trace for u, and a unique solution $\tilde{u}$ for this trace. By uniqueness (Theorem 4.3.2) we obtain the result. □

4.5 Asymptotic behavior (slow diffusion)

In this section we shall show that the exceptional solution $\alpha(x,t)$ of (4.0.1) with $\varphi \in \mathcal{S}_a$ is a universal attractor.

Theorem 4.5.1. *Let $\varphi \in \mathcal{S}_a$. Assume that*

$$\lim_{h \downarrow 0} \frac{\varphi(hu)}{\varphi(h)} = \psi(u)$$

uniformly on compact subsets of $[0, \infty)$. *Then if $u \not\equiv 0$ is a strong solution of the IDP* (4.0.1) *in $D \times (0, \infty)$ then*

$$\lim_{t \uparrow \infty} \frac{u(x,t)}{\alpha_\varphi(x,t)} = 1$$

uniformly for $x \in D$.

Proof. Define $\Lambda_\varphi(u)$ the inverse function of $\frac{\varphi(u)}{u}$ on $(0, \infty)$. For $s > 0$ let

$$\psi_s(u) \equiv \frac{\varphi(\Lambda_\varphi(1/s)u)}{\varphi(\Lambda_\varphi(1/s))}.$$

It is not hard to check that $\psi_s \in \mathcal{S}_a$ and $\Lambda_\varphi(r) \to 0$ as $r \to 0$.

Define $v_s(x,t) = \frac{u(x,st)}{\Lambda_\varphi(1/s)}$. Thus v_s is a strong solution of the IDP for the equation $\frac{\partial v}{\partial t} = \Delta \psi_s(v)$. We claim that

$$\lim_{s \to 0} v_s(x,t) = \alpha_\psi(x,t), \tag{4.5.1}$$

where $\alpha_\psi(\cdot,\cdot)$ is the corresponding exceptional function for $\psi \in \mathcal{S}_a$.

From the existence proof of α (Theorem 4.2.1) we have that for any $\theta \in \mathcal{S}_a$, $\alpha_\theta(x,t) \le C/t^\alpha$ where C and α depend only on a.

Note also that $\psi_s \to \psi$ as $s \to \infty$ uniformly on compact sets by assumption.

Thus if $\{s_j\}$ is a sequence going to $+\infty$, there exists a subsequence $\{s_{j_k}\}$ such that $v_{s_{j_k}} \to v$ uniformly on compact subsets of $\bar{D} \times (0,\infty)$ (Note that strong solutions of the IDP for $\theta \in \mathcal{S}_a$ are equicontinuous). Moreover v is a strong solution of the IDP for $\frac{\partial v}{\partial t} = \Delta\psi(v)$. It remains to show that $v = \alpha_\psi$.

Let $w(x,t) = Gu(x,t)$ and $h(x) = \lim_{t\downarrow 0} w(x,t)$ where

$$h(x) = \int_D G(x,y)d\mu(y) + \int_{\partial D} K(x,Q)d\lambda(Q)$$

where either μ or λ is not 0. Hence $h(x) \ge c\delta(x)$ (if $\mu \ne 0$ it is easy to prove it; when $\mu = 0$ we use the Hopf maximum principle).

Let now $f \in C_0(D)$ and denote by $u_{j_k,f}$ the strong solution of the IDP corresponding to $\psi_{s_{j_k}}$ and f.

We claim that for j_k large $Gv_{s_{j_k}} \ge w_{j_k,f} = Gu_{j_k,f}$. By Lemma 4.1.3 it suffices to check it at $t = 0$. But $w_{j_k,f}(x,0) = Gf(x)$ since $u_{j_k,f}$ is continuous up to $t = 0$, and

$$\lim_{t\downarrow 0} Gv_{s_{j_k}}(x,t) = \lim_{t\downarrow 0} \frac{w(x,s_{j_k}t)}{\Lambda_\varphi(1/s_{j_k})} = \frac{h(x)}{\Lambda_\varphi(1/s_{j_k})}.$$

Since $Gf(x) \le C\,\delta(x)$ we establish the claim.

As $j_k \to \infty$, $\psi_{s_{j_k}} \to \psi$, and hence $u_{j_k,f} \to u_f$ uniformly on compact subsets of $\bar{D} \times [0,\infty)$, where u_f denotes the strong solution of the IDP for ψ with initial f. Hence, if $w_f = Gu_f$ then $Gv \ge w_f$. Therefore $Gv \ge G\alpha_\psi$, which implies that $v = \alpha_\psi$ and the claim (4.5.1) is proven.

From Theorem 4.2.1 it follows that

$$\alpha_{\psi_s}(x,t) = \frac{\alpha_\psi(x,st)}{\Lambda_\varphi(1/s)}. \tag{4.5.2}$$

Hence

$$\lim_{s\to\infty} \frac{u(x,st)}{\Lambda_\varphi(1/s)\alpha_\psi(x,t)} = 1 = \lim_{s\to\infty} \frac{u(x,st)\alpha_{\psi_s}(x,st)}{\Lambda_\varphi(1/s)\alpha_{\psi_s}(x,st)\alpha_\psi(x,t)}$$

$$= \lim_{s\to\infty} \frac{u(x,st)\alpha_{\psi_s}(x,t)}{\alpha_\varphi(x,st)\alpha_\psi(x,t)}.$$

However, we claim that

$$\lim_{s\to\infty} \frac{\alpha_{\psi_s}(x,t)}{\alpha_\psi(x,t)} = 1$$

which would complete the proof of the theorem. But (4.5.2) and the proof that $\lim_{s\to\infty} v_s(x,t) = \alpha_\psi(x,t)$ can be combined to establish the claim. □

Remark. In the case of the porous medium equation $\varphi(u) = u^m, m > 1$, the asymptotic behavior of solutions to IDP can be described in a more precise way. The exceptional solution takes the special form

$$U(x,t) = t^{-\alpha} f(x), \quad \alpha = \frac{1}{m-1}$$

where $f(x)$ is a solution of the stationary elliptic problem

$$\Delta f^m + \alpha f = 0.$$

In this case it can be shown, see [128], that each non-negative weak solution u of the IDP (see [128] for the exact definition of weak solution) satisfies

$$\lim_{t\to\infty} t^\alpha |u(x,t) - U(x,t)| = \lim_{t\to\infty} |t^\alpha u(x,t) - f(x)| = 0.$$

This result is optimal in the sense that the exponent α cannot be improved and the profile function f is uniquely determined. However, one can improve the above result by estimating the error $t^\alpha u(x,t) - f(x)$. We refer the reader to [128] and the references therein for all the details.

4.6 A priori estimates (fast diffusion)

In this section we shall prove two a priori estimates for strong solutions to the IDP (4.0.1) in the fast diffusion, pure power case $\varphi(u) = u^m$, $0 < m < 1$. The first lemma, which provides a local L^1 bound on u, holds for all $0 < m < 1$. However, the second lemma, which provides an L^∞ bound on u, holds only in the range of exponents $m_1 < m < 1$, with $m_1 = (n-1)_+/n$. The new critical exponent m_1 which enters here will be discussed later.

Lemma 4.6.1. *Let u be a strong solution of the IDP (4.0.1) with $\varphi(u) = u^m$, $0 < m < 1$. Then*

$$\sup_{t>0} \int_D u(x,t)\, \delta(x)\, dx < \infty$$

with $\delta(x) = \operatorname{dist}(x, \partial D)$.

Proof. Let $e(x)$ be a ground state for $-\Delta$ on D, i.e. $\Delta e = -\lambda\, e$ in D, for some $\lambda > 0$, $e = 0$ on ∂D, $e > 0$ in D, $e \in C^\infty(\bar{D})$. Consider $M(t) = \int_D u(x,t)\, e(x)\, dx$. Clearly $dM(t)/dt \le 0$, so that for any t_0 we have

$$\sup_{t>t_0} M(t) \le M(t_0).$$

Because of this, if there exists $t_n \to 0$ such that $M(t_n) \to 0$, $M \equiv 0$ and so is u. Thus, it remains to show that if for some $\delta > 0$ there exists an $\varepsilon > 0$ such that $M(t) \ge \varepsilon$, for

$0 < t \le \delta$, then $\sup_{0<t<\delta} M(t) < \infty$. But

$$\frac{dM(t)}{dt} = -\lambda \int_D u^m(x,t)\, e(x)\, dx$$

and

$$\int_D u^m(x,t)\, e(x)\, dx \le A \left(\int_D u(x,t)\, e(x)\, dx \right)^m$$

by Hölder's inequality, since $m < 1$. Thus, we have

$$0 \ge \frac{dM(t)}{dt} \ge -(A\,\lambda)\, M(t)^m \quad \text{for } 0 < t \le \delta,$$

and also $M(t) \ge \varepsilon$, for $0 < t \le \delta$. An easy ODE comparison argument now gives the result. □

Lemma 4.6.2. *Let u be a strong solution of the IDP (4.0.1), with $\varphi(u) = u^m$, $m_1 < m < 1$, $m_1 = (n-1)_+/n$. Then, for all $0 < t_0 < t < t_1 < \infty$ we have*

$$\|u\|_{L^\infty(\bar{D}\times[t_0,t_1])} \le C(M, m, n, t_0, t_1)$$

with $M = \sup_{t>0} \int_D u(x,t)\, \delta(x)\, dx$.

Proof. We first observe that Theorem 1.4.3 in Chapter 1 gives, for $R^2 < t$, $0 < s < t$, the bound

$$\sup_{|x-x_0|<R} u(x,t) \le C \left\{ \frac{1}{t^\theta} \left[\int_{|x-x_0|<4R} u(x,s)\, dx \right]^{\frac{2\theta}{n}} + \left(\frac{t}{R^2} \right)^{\frac{1}{1-m}} \right\}$$

where $\theta^{-1} = 2/n - (1-m)$. If we choose $R \approx \delta(x_0)$, we obtain the bound

$$\sup_{|x-x_0|<\delta(x_0)} u(x,t) \le C_t \left\{ M^{\frac{2\theta}{n}}\, \delta^{-\frac{2\theta}{n}} + \delta(x_0)^{-\frac{2}{1-m}} \right\}.$$

If $m > m_1$, we see that $\frac{2\theta}{n} \le \frac{2}{1-m}$, and hence

$$u(x_0, t) \le C(M, m, n, t_0, t_1)\; \delta(x_0)^{-\frac{2}{1-m}}, \qquad t \in [t_0, t_1]. \tag{4.6.1}$$

Next, observe that the proof of Theorem 1.4.1 in Chapter 1 still applies to subsolutions and thus, to u extended by 0 outside of $\bar{D} \times (0, \infty)$. Hence, by the remark after the proof of Theorem 1.4.1 in Chapter 1 we see that if $q > (1-m)\,n/2$ we have

$$\|u\|_{L^\infty(\bar{D}\times[t_0,t_1])} \le C \left[\iint_{\bar{D}\times[t_0,t_1]} u^q + 1 \right]^\sigma.$$

We next set $q = \varepsilon + m$, and note that if $m > m_1$, we can choose $0 < \varepsilon < (1-m)/2$ so that $q > (1-m)\,n/2$. We shall now establish, for this choice of ε, that

$$\iint_{\bar{D}\times[t_0,t_1]} u^q \le C(M, m, n, t_0, t_1)$$

concluding the proof of the lemma. Observe that by (4.6.1),

$$u^{\varepsilon}(x,t) \le C(M,m,n,t_0,t_1)\,\delta(x)^{-\frac{2\varepsilon}{1-m}}.$$

Let $\alpha = 2\varepsilon/(1-m)$, so that $0<\alpha<1$. Note that standard estimates on the Green's function yield that $G(\delta^{-\alpha}) \le C_\alpha\,\delta$. Hence

$$\begin{aligned} C\,M &\ge \int_D [\,u(x,t_0/2) - u(x,2t_1)\,]\,G(\delta^{-\alpha})\,dx \\ &= \int_{t_0/2}^{2t_1}\int_D u^m\,\delta^{-\alpha} \\ &\ge C\int_{t_0/2}^{2t_1}\int_D u^{m+\varepsilon} \end{aligned}$$

as required. □

4.7 The trace and uniqueness (fast diffusion)

In this section we shall establish the existence of initial trace and the uniqueness for strong solutions to the IDP (4.0.1) in the fast diffusion, pure power case $\varphi(u) = u^m$, $0<m<1$. Indeed, the proof of Theorem 4.3.1 combined with Lemma 4.6.1 gives:

Theorem 4.7.1. *Let $u(x,t)$ be a strong solution of the IDP (4.0.1), with $\varphi(u) = u^m$, $0<m<1$. Then there exist non-negative Borel measures μ and λ on D and ∂D respectively with*

$$\int_D \delta(x)\,d\mu(x) < \infty \quad \text{and} \quad \int_{\partial D} d\lambda < \infty$$

such that for any $\eta \in C_0^\infty(\mathbb{R}^n)$ with $\eta|_{\partial D} = 0$,

$$\lim_{t\downarrow 0}\int_D u(x,t)\eta(x)dx = \int_D \eta(x)\,d\mu(x) - \int_{\partial D}\frac{\partial\eta}{\partial n}(Q)\,d\lambda(Q)$$

where $\partial/\partial n$ denotes the exterior normal derivative to ∂D.

Also, in analogy with Theorem 4.3.2, with the obvious modification of the proof, we have:

Theorem 4.7.2. *Suppose u_1, u_2 are two strong solutions of the IDP (4.0.1), with $\varphi(u) = u^m$, $0<m<1$. Assume that*

$$\lim_{t\downarrow 0}\int_D u_1(x,t)\,\eta(x)\,dx = \lim_{t\downarrow 0}\int_D u_2(x,t)\,\eta(x)\,dx$$

for any $\eta \in C_0^\infty(\mathbb{R}^n)$, $\eta \equiv 0$ on ∂D. Then $u_1 \equiv u_2$.

4.8 Existence of solutions (fast diffusion $m_1 < m < 1$)

This section is devoted in establishing the existence of strong solutions of the IDP (4.0.1) in the fast diffusion case $\varphi(u) = u^m$, $m_1 < m < 1$. This result is due to Chasseigne and Vazquez [37] but will present here a different proof, based on the a priori estimate Lemma 4.6.2.

Theorem 4.8.1. *Let $\varphi = u^m$, $m_1 < m < 1$, $m_1 = (n-1)_+/n$. Given a pair of measures μ on D and λ on ∂D with $\mu \geq 0$, $\lambda \geq 0$, $\int_D \delta(x)\, d\mu(x) < \infty$ and $\int_{\partial D} d\lambda < \infty$, there exists a unique strong solution u of the IDP (4.0.1) in $D \times (0, \infty)$ such that for any $\eta \in C^\infty(\mathbb{R}^n)$, $\eta \equiv 0$ on ∂D*

$$\lim_{t \downarrow 0} \int_D u(x,t)\, \eta(x) dx = \int_D \eta d\mu - \int_{\partial D} \frac{\partial \eta}{\partial n}\, d\lambda$$

where $\partial/\partial n$ denotes the exterior normal derivative to ∂D.

Proof. The proof is analogous to the one of Theorem 4.4.1 and proceeds in steps. We shall only give a sketch of the proof emphasizing only the parts of the proof which are different from those in the proof of Theorem 4.4.1.

Step 1. Assume $\lambda \equiv 0$ and supp $\mu \subset K \subset\subset D$. Denote by u_f the solution of the BVP (see Section 1.6 of Chapter 1)

$$\begin{aligned} u_t &= \Delta\varphi(u) && \text{in } D \times (0,\infty), \\ u &= 0 && \text{on } \partial D \times (0,\infty), \\ u(x,0) &= f(x) && x \in D, \end{aligned}$$

where $f \in C_0^\infty(D)$, $f \geq 0$. Thus $u_f \in C(\bar{D} \times [0,\infty))$.

Claims.

(i) $M = \sup_{t>0} \int_D u_f(x,t) dx \leq \int_D f(x)\, dx$.

(ii) $\sup_{t>0} \int_D u_f(x,t)\, e(x) dx \leq \int_D f(x)\, e(x)\, dx$, where e is as in Lemma 4.6.1.

(iii) If $\eta \in C^\infty(\mathbb{R}^n)$, $\eta \equiv 0$ on ∂D, then for $t \in (0,1)$

$$\left| \int_D [u_f(x,t) - f(x)]\, \eta(x)\, dx \right| \leq C_{M,\eta}\, t.$$

Proof of Claims. (i) follows as in the proof of Theorem 4.4.1. (ii) follows from the proof of Lemma 4.6.1. For (iii) we observe that

$$\int_D [u_f(x,t) - f(x)]\, \eta(x)\, dx = \int_D u_f^m\, \Delta\eta\, dx\, ds \leq \int_0^t \sup|\Delta\eta|\, M^m\, ds$$

and the claim follows.

Now pick $\{f_j\} \subseteq C_0^\infty(D)$, $f_j \geq 0$ such that $\int_D f_j dx \leq \int_D d\mu$, supp $f_j \subset \overline{K} \subset D$, and $f_j\, dx \to d\mu$ weakly. Let $u_j = u_{f_j}$. Because of Lemma 4.6.2 and the equiconinuity result in Theorem 1.5.1 in Chapter 1, passing to a subsequence $\{u_j\}$ converges uniformly in $\overline{D} \times [s, \tau]$ to a strong solution u of the IDP (4.0.1), for any $0 < s < \tau < \infty$. As in the proof of Theorem 4.4.1 we see that u has trace μ, completing the proof of Step 1.

Step 2. General case. We again introduce the superharmonic function

$$h(x) = \int_D G(x, y) d\mu(y) + \int_{\partial D} K(x, Q) d\lambda(Q)$$

with $h \geq 0$, $\int_D h(x)dx < \infty$ and the sequence $\{h_j\}$ as in the proof of Step 2 in Theorem 4.4.1. Let u_j be the solution corresponding to μ_j, constructed in Step 1. Denote by $w_j(x,t)$ the Green's potential of u_j, so that $\lim_{t\downarrow 0} w_j(x,t) = h_j(x)$. We need to observe now that since $h_j \uparrow h$ and $h \in L^1(D)$, there exists x_0 so that $h_j(x_0) \leq h(x_0) < \infty$. Thus, $\sup_j \int_D e(x)\, d\mu_j < +\infty$, and hence

$$\sup_{t>0} \sup_j \int_D e(x)\, u_j(x,t)\, dx < \infty.$$

By Lemmas 4.6.1 and 4.6.2 and the equicontinuity result in Theorem 1.5.1, a subsequence of $\{u_j\}$ converges in $\overline{D} \times [s, \tau]$ to a strong solution u of the IDP (4.0.1). To check that u has the right trace we only need to check that for $w(x,t) = Gu(x,t)$ the limit $\lim_{t\downarrow 0} w(x,t) = h(x)$. As in Theorem 4.4.1, $w(x,t) \leq h(x)$, for all $t > 0$. The proof of Theorem 4.4.1 also gives that $w(x,t) \geq w_j(x,t)$, which concludes the proof. □

Remark. The existence theorem for strong solutions fails when $0 < m \leq m_1$, as pointed out in [37]. The unique solution in $B_1 \times (0, T)$, with initial data $(T/|x|^2)^{\frac{1}{1-m}}$ (obtained as a limit of classical solutions) is not bounded near $x = 0$, for $0 < m \leq m_c$. Moreover, for the case $m = m_1$, if f solves the elliptic eigenvalue problem $-\Delta_{S^{n-1}} f^m = C_n f^m$ in S^{n-1} and we impose the boundary condition $f(\theta) = 0$ on ∂S^{n-1}, where $\theta = x/|x|$, then $u(x,t) = |x|^{\frac{-2}{1-m}} f(\theta)$ is a stationary solution with $D = \mathbb{R}^n_+$, which is a limit of classical solutions. If we take $f(\theta)^m = x_1/|x|$, $C_n = n-1$, one obtains the solution

$$u(x,t) = k \left(\frac{x_1}{|x|^n} \right)^{\frac{n+1}{n-1}}, \quad k > 0$$

which is zero at $\partial \mathbb{R}^n_+ \setminus \{0\}$ but it is not bounded near $x = 0$ (see [37]). Corresponding solutions for $0 < m < m_1$ are also found in [37].

4.9 Further results and open problems

We shall present in this last section a collection of further results concerning strong solutions of equation $u_t = \Delta\varphi(u)$ in an infinite cylinder $D \times (0, \infty)$.

1. Extinction in finite time. In [37] it is shown that if $0 < m < 1$ and u is a classical solution to the IDP (4.0.1), with bounded initial data, it must vanish identically in finite time. Moreover, it is shown that for $m_1 < m < 1$, all strong solutions of the IDP vanish identically in finite time T and are smooth in $\bar{D} \times (0, T)$.

2. The ranges $m_c < m \le m_1$, $0 < m \le m_c$. In [37] the class $\mathcal{E}_0$ consisting of non-negative weak solutions of the IDP (4.0.1) (i.e. for which (4.0.2) holds), such that $u \in L^1_{\mathrm{loc}}((0,T); L^1(D, \delta(x)\,dx))$, $\delta(x) = \mathrm{dist}(x, \partial D)$, and $u^m \in L^1_{\mathrm{loc}}((0,T); L^1(D))$, for all $0 < T < \infty$, is introduced. It is shown that for $0 < m < 1$, $u \in \mathcal{E}_0$, the initial traces μ, λ exist, that $\sup_{t>0} \int_D u(x,t)\,\delta(x)\,dx < \infty$ and $\int_0^T \int_D u^m(x,t)\,dx\,dt < \infty$. It is also shown that μ does not charge sets of zero $C_{2,\frac{1}{1-m}}$ capacity in D (a vacuous condition for $m > m_c$) and that λ does not charge sets of zero $C_{2m,\frac{1}{1-m}}$ capacity on ∂D (a vacuous condition for $m > m_1$). It is also shown that given μ with $\int_D \delta(x)\,d\mu(x) < \infty$, which does not charge sets of zero $C_{2,\frac{1}{1-m}}$ capacity in D and λ with $\int_{\partial D} d\lambda < \infty$, which does not charge sets of zero $C_{1,\frac{1}{1-m}}$ capacity on ∂D, there exists $u \in \mathcal{E}_0$ with that trace. Moreover, in the range $m_c < m \le m_1$, when λ does not charge sets of zero $C_{1,\frac{1}{1-m}}$ capacity on ∂D the solution is unique in $\mathcal{E}_0$. In addition, if $m \in (m_c, m_1)$, λ does not charge sets of zero $C_{2m,\frac{1}{1-m}}$ capacity on ∂D and $\int_D \delta(x)^\alpha\,d\mu(x) < \infty$, $\alpha < \alpha_0 = -n + 2/(1-m)$, then u is a strong solution of the IDP.

3. Extinction for solutions in $\mathcal{E}_0$. If $m_1 < m < 1$, $u \in \mathcal{E}_0$, then there exists a time $0 < T \le \infty$ such that $u > 0$ is smooth in $D \times (0, T)$, and $u \equiv 0$, for $t \ge T$. If $m_c < m < m_1$, extinction in finite time happens for $u \in \mathcal{E}_0$ with trace μ, λ, with λ satisfying the capacity condition and $\int_D \delta(x)^\alpha\,d\mu(x) < \infty$, $\alpha < \alpha_0 = -n+2/(1-m)$. Also, examples are constructed in [37] which show that for $0 < m \le m_c$ not all u in $\mathcal{E}_0$ with data in $L^1(D)$ vanish in finite time.

4. Very hot spots. In [37] solutions to the IDP in a class more general than $\mathcal{E}_0$ are constructed. These are continuous as functions into the extended reals.

5. The initial Neumann problem in an infinite cylinder. Sze [121] studied the initial Neumann problem

$$\left.\begin{aligned} u_t &= \Delta\varphi(u) && \text{in } \Omega = D \times (0,\infty), \\ \partial u/\partial \nu &= 0 && \text{on } \partial D \times (0,\infty), \end{aligned}\right\} \tag{4.9.1}$$

where $D \subseteq \mathbb{R}^n$ is an open bounded domain with smooth boundary. The nonlinearity φ belongs to the class of $\mathcal{S}_a$, as it was defined at the beginning of this chapter, however it is assumed in [121] that the growth condition

$$1 + a \le \frac{u\,\varphi'(u)}{\varphi(u)} \le a^{-1}.$$

holds for all $0 < u < \infty$ (not only for $1 < u < \infty$). Let us denote by $\tilde{\mathcal{S}}_a$ the class of such nonlinearities. The best known example of such φ is the porous medium equation $\varphi(u) = u^m, m > 1$.

We say that u is a *strong solution* of the initial Neumann problem (INP) (4.9.1) if u is a non-negative continuous function in $\bar{D} \times (0, \infty)$ and for all $\eta \in C^\infty(\bar{D} \times [0, \infty))$ with $\partial\eta/\partial\nu = 0$ on $\partial D \times [t_1, t_2]$, $0 < t_1 < t_2$, we have

$$\begin{aligned} &\int_D \int_{t_1}^{t_2} \left(\varphi(u)\Delta\eta + u\frac{\partial\eta}{\partial t}\right) dxdt \\ &\qquad = \int_D u(x,\tau_2)\,\eta(x,\tau_2)dx - \int_D u(x,\tau_1)\,\eta(x,\tau_1)dx. \end{aligned} \tag{4.9.2}$$

The main result in [121] is that all solutions of the INP (4.9.1) are in one-to-one correspondence with non-negative finite Borel measures on $\bar{D}$. More precisely, if u is a solution of the INP (4.9.1), then it has a trace as $t \downarrow 0$, i.e., there is a non-negative finite Borel measure μ on $\bar{D}$ such that

$$\lim_{t\downarrow 0} \int_D u(x,t)\,\eta(x)\,dx = \int_{\bar{D}} \eta(x)\,d\mu(x)$$

whenever

$$\eta \in C^\infty(\bar{D}) \quad \text{and} \quad \frac{\partial\eta}{\partial\nu}|_{\partial D} \equiv 0.$$

Moreover, the trace determines the solution uniquely and vice versa. Also, this is the correct class of traces, i.e., if μ is a non-negative finite Borel measure on $\bar{D}$, then there exists a unique solution u with initial trace μ. Finally, it was shown in [121] that solutions can be uniquely extended to be solutions in $D \times (t_1, \infty)$.

We note that the theory for the INP is quite similar to the theory for the IDP which was described in detail in this chapter. The only difference is the absence of the "exceptional solution" or "friendly giant" in the case of the INP. Since the proofs of the above stated results are similar to the ones for the IDP which were given in detail in Sections 4.1–4.5 of this chapter, we shall not present them here, referring the reader to [121].

Chapter 5
Weak solutions

This chapter is devoted to the study of the regularity properties of weak solutions to the porous medium equation

$$\frac{\partial u}{\partial t} = \Delta u^m, \quad m > 1$$

showing that weak solutions are continuous almost everywhere. In the previous chapters it was crucial to deal with weak solutions that were a priori known to be continuous. Our main results in this chapter remove this assumption, showing that weak solutions are indeed continuous. In the first section we shall restrict ourselves to weak solutions which are zero in a weak sense on the lateral part of the boundary, establishing the continuity of such solutions. In the second section we shall consider the purely local version of this result.

5.1 Weak solutions of the porous medium equation in a cylinder

Let D be an open bounded set of $\mathbb{R}^n$ with smooth boundary. Let M denote the class

$$M = \left\{\eta \in C^\infty(\mathbb{R}^n \times (0,\infty)) : \eta(\cdot, t) \in C_0^\infty(\mathbb{R}^n),\ \eta \equiv 0 \text{ on } \partial D \times (0,\infty)\right\}.$$

Definition. We say that $u(x,t)$ is a weak solution of the initial Dirichlet problem (IDP)

$$\frac{\partial u}{\partial t} = \Delta u^m, \quad m > 1, \ \text{ in } D \times (0,\infty) \tag{5.1.1}$$

if

(i) $u \geq 0$,

(ii) u satisfies

$$\iint_{D\times(a,b)} u^m \, dxdt < \infty$$

for any $0 < a < b < \infty$,

(iii) for any $\eta \in M$, u satisfies the integral identity

$$\iint_{D\times(0,\infty)} \left[u \frac{\partial \eta}{\partial t} + u^m \Delta\eta\right] dxdt = 0.$$

Observe that iii) tells us that $u \equiv 0$ on $\partial D \times (0, \infty)$ in the weak sense.

Our main result in this section is the following result from [50].

Theorem 5.1.1. *If u is a weak solution of the IDP (5.1.1) then there exists u^* continuous on $\overline{D} \times (0, \infty)$ such that $u = u^*$ a.e. in $D \times (0, \infty)$. Moreover, $u^* = 0$ on $\partial D \times (0, \infty)$.*

Proof. For the proof of Theorem 5.1.1 we shall follow three steps:

Step 1. *Existence of trace.* Define $w = Gu$, i.e. the Green's potential of u in $\Omega = D \times (0, \infty)$. Thus $w \geq 0$ and

$$\int_a^b \int_D w(x, t)\, dxdt < \infty.$$

The above estimate follows from the relation between the Green's potential $G(\cdot, \cdot)$ and the distance $\delta(\cdot)$ to the boundary.

We claim that

$$\iint_\Omega \left[w \frac{\partial \eta}{\partial t} + \eta\, u^m \right] dxdt = 0 \quad \text{for all } \eta \in M. \tag{5.1.2}$$

Indeed, if $\eta \in M$ then $G\Delta\eta = -\eta$. Thus for $\theta \in C_0^\infty(\Omega)$, $\eta = \eta_\theta \equiv G\theta \in M$ and

$$\begin{aligned} \iint_\Omega w\, \frac{\partial \theta}{\partial t} &= \iint w\, (-\Delta) \left(\frac{\partial \eta}{\partial t} \right) \\ &= \iint u\, \frac{\partial \eta}{\partial t} \\ &= -\iint u^m\, \Delta\eta \\ &= \iint_\Omega u^m\, \theta. \end{aligned}$$

Therefore, $\partial w / \partial t = -u^m$ in $D'(\Omega)$ and the claim (5.1.2) is proved.

Consequently if $\tau > 0$, $\tilde{\theta} \in C_0^\infty(D)$, and $\gamma \in C_0^\infty(\mathbb{R})$ with $\gamma(\tau) = 0$ an approximation argument shows that

$$\iint_{D \times (\tau, \infty)} \tilde{\theta}(x) \left[\gamma(t)\, (u(x, t))^m - \gamma'(t)\, w(x, t) \right] dxdt = 0. \tag{5.1.3}$$

Next we want to define the trace for $w(x, t)$ at any time τ. Pick $\tilde{\gamma} \in C_0^\infty(\mathbb{R})$ such that $\tilde{\gamma}(\tau) = 1$ and define

$$\int_D \tilde{\theta}\, dv_\tau = \iint_{D \times (\tau, \infty)} \tilde{\theta}(x) \left[\gamma(t)\, (u(x, t))^m - \gamma'(t)\, w(x, t) \right] dxdt \tag{5.1.4}$$

for any $\tilde{\theta} \in C_0^\infty(D)$. From (5.1.3) we have that v_τ is well defined, i.e., it does not depend on the choice of γ.

Lemma 5.1.2. $\{v_\tau\}_{\tau>0}$ *defined in* (5.1.4) *is a non-negative weakly continuous family of measures with finite positiveness, i.e.,*

$$\int_D dv_\tau < \infty.$$

Moreover $\Delta v_\tau \le 0$ *in* $D'(D)$.

Proof. It is clear from (5.1.4) that v_τ is a measure. In fact, it is a positive measure: for $\varepsilon > 0$ choose $\gamma_\varepsilon \in C_0^\infty(\mathbb{R})$ dying off at infinity such that $\gamma_\varepsilon'(t) \ge 0$ for $t \in (\tau - \varepsilon, \tau)$, $\gamma_\varepsilon(t) = 0$ for $t \le \tau - \varepsilon$, and $\gamma_\varepsilon(t) = 1$ for $t \ge \tau$. Then from (5.1.3) - (5.1.4) we obtain that

$$\begin{aligned} -\int \tilde{\theta}(x)\, dv_\tau &= \iint_{D\times(\tau-\varepsilon,\tau)} \tilde{\theta}(x)\, [\, \gamma_\varepsilon(t)\, (u(x,t))^m - \gamma_\varepsilon'(t)\, w(x,t)\,]\, dxdt \\ &\le \iint_{D\times(\tau-\varepsilon,\tau)} \tilde{\theta}(x)\, \gamma_\varepsilon(t)\, (u(x,t))^m\, dxdt \to 0 \quad \text{as } \varepsilon \downarrow 0. \end{aligned}$$

Therefore $\int_D \tilde{\theta}\, dv_\tau \ge 0$, showing that $dv_\tau \ge 0$.

We also have that

$$\int_D dv_\tau < \infty \tag{5.1.5}$$

which can be readily seen by taking $\{\tilde{\theta}_j\}_1^\infty$ an increasing sequence in $C_0^\infty(D)$ converging to χ_D.

Also, a direct computation shows that

$$\tau \to v_\tau \quad \text{is weakly continuous.} \tag{5.1.6}$$

Finally, to see that $\Delta v_\tau \le 0$, let $\gamma \in C_0^\infty(\mathbb{R})$ with $\gamma(\tau) = 1$ and $l_j \in C_0^\infty(\mathbb{R})$ such that $l_j' \ge 0$ and

$$l_j(t) = \begin{cases} 0 & \text{for } t < \tau, \\ 1 & \text{for } t > \tau + 1/j. \end{cases}$$

Then, for $\tilde{\theta} \in C_0^\infty(D)$ such that $\tilde{\theta} \ge 0$ we have that

$$\begin{aligned} \int_D \Delta\tilde{\theta}\, dv_\tau &= \iint_{D\times(\tau,\infty)} \Delta\tilde{\theta}(x)\, [\, \gamma(t)\ \ (u(x,t))^m - \gamma'(t)\, w(x,t)\,]\, dxdt \\ &= \lim_{j\to\infty} \iint_{D\times(\tau,\infty)} \Delta\tilde{\theta}(x)\, l_j(t)\, [\, \gamma(t)\, (u(x,t))^m - \gamma'(t)\, w(x,t)\,]\, dxdt \\ &= \lim_{j\to\infty} \iint_{D\times(\tau,\infty)} \left[-u\tilde{\theta}\, l_j \gamma' - u\tilde{\theta}\, l_j' \gamma + u\tilde{\theta}\, l_j \gamma' \right] dxdt \le 0 \end{aligned}$$

since $l_j' \ge 0$, finishing the proof of the lemma. □

We recall that by Potential Theory it follows: if $\int_D d\mu < \infty$, $\mu > 0$ and $\Delta\mu \le 0$, then μ is absolutely continuous.

Step 2. Principle procedure. We begin by defining a convenient approximation of the Green's function. For $f \in C_0^\infty(D)$, let u_f be the solution of the initial Dirichlet problem for the heat equation, i.e. u_f solves the equation

$$\left.\begin{aligned} u_t - \Delta u &= 0 && \text{in } D \times (0,\infty), \\ u(x,0) &= f(x) && \text{for } x \in D, \\ u(x,t) &= 0 && \text{for } x \in \partial D,\ t > 0. \end{aligned}\right\} \tag{5.1.7}$$

The solution u_f can be written as

$$H_t\, f(x) = u_f(x,t) = \int_D H_t(x,y)\, f(y)\, dy$$

where the kernel $H_t(\cdot,\cdot)$ satisfies

(i) $H_t(x,y) \ge 0$ for all $(x,y) \in D \times D$;

(ii) $H_t(x,y) = H_t(y,x)$;

(iii) $\partial_t H_t(x,y) - \Delta_x H_t(x,y) = 0$;

(iv) $Gf(x) =$ Green's function of $f = \int_0^\infty u_f(x,t)dt$.

More precisely, if f_j is the j-eigenfunction of the Laplacian, i.e.,

$$\begin{aligned} -\Delta f_j &= \lambda_j f_j && \text{in } D, \\ f_j &= 0 && \text{in } \partial D, \end{aligned}$$

then

$$H_t(x,y) = \sum e^{-\lambda_j t} f_j(x)\, f_j(y).$$

We denote by $G_k f$ the operator defined by

$$G_k f(x) = \int_{2^{-k}}^\infty u_f(x,t)\, dt \tag{5.1.8}$$

and let $G_k(x,y)$ denote the corresponding kernel, i.e.

$$G_k(x,y) = \int_{2^{-k}}^\infty H_t(x,y)\, dt = \int_{2^{-k}}^\infty \sum e^{-\lambda_j t}\, f_j(x)\, f_j(y)\, dt.$$

Thus

(i) $G_k(x,y) = G_k(y,x)$, $G_k \in C_0^\infty(D \times D)$ and $G_k \ge 0$;

(ii) $G_k(x,y) \uparrow G(x,y)$;

(iii) $\Delta_x G_k(x, y) \le 0$ (superharmonic).

Defining $p_k(x, t) = G_k u$, we have

(1) $p_{k+1} \ge p_k$;

(2) $\partial_t p_k \le 0$ in the distribution sense.

The first inequality follows since G_k is monotonically increasing in k. To prove 2) we pick $\eta \in C_0^\infty(D)$ with $\eta \ge 0$ so that

$$\iint \frac{\partial \eta}{\partial t}\, p_k\, dxdt = \iint \frac{\partial (G_k \eta)}{\partial t}\, u = -\iint \Delta(G_k \eta)\, u^m \ge 0.$$

Next we take $\lambda \in C_0^\infty([1/2, 1])$ such that $\lambda \ge 0$ and $\int \lambda(t)dt = 1$ and define

$$w_k(x, t) = \int p_k(x, t + s2^{-k})\, \lambda(s)\, ds \tag{5.1.9}$$

i.e., the regularization in time with width 2^{-k}.

Lemma 5.1.3. *The functions w_k are smooth and converge to w in the distribution sense. Moreover, for $k = 1, 2, \dots,$*

(i) $w_k \le w_{k+1}$,

(ii) $\partial_t w_k \le 0$,

(iii) $\Delta w_k \le 0$,

(iv) $\partial_t w_k \le \psi(\Delta w_k)$ *where* $\psi(s) = \mathrm{sgn}(s)\, |s|^m$.

In addition, for every $\tau > 0$ there exists $f_\tau \in L^1(D)$, $f_\tau \ge 0$ such that $w_k(x, t) \le G f_\tau$ for $t > \tau$ and for all $0 < a < b < \infty$

$$\iint_{D\times(a,b)} \left| \frac{\partial w_k}{\partial t} - \frac{\partial w}{\partial t} \right| dxdt \to 0$$

as $k \to \infty$.

Proof. By definition

$$\begin{aligned} w_k(x, t) = \int p_k(x, t + s2^{-k})\, \lambda(s)\, ds &\le \int p_{k+1}(x, t + s2^{-k})\, \lambda(s)\, ds \\ &\le \int p_{k+1}(x, t + s2^{-(k+1)})\, \lambda(s)\, ds = w_{k+1}(x, t) \end{aligned}$$

because $\partial_t p_k(x, t) \le 0$. The proof of $\partial_t w_k(x, t) \le 0$ and $\Delta w_k \le 0$ are similar, and will be omitted. With the above notation define the operator P_k for a function q defined on $\Omega = D \times (0, \infty)$ by

$$P_k q(x, t) = \iint H_{2^{-k}}(x, y)\, \lambda(s)\, q(y, t + s2^{-k})\, dyds \tag{5.1.10}$$

with the notation $w_k = P_k w$, and

$$\psi(\Delta w_k) - \frac{\partial w_k}{\partial t} = P_k u^m - (P_k u)^m \geq 0$$

by the fact that P_k is given by a non-negative kernel with total integral less or equal than 1 and Jensen's inequality. Notice that

$$\frac{\partial (P_k q)}{\partial t} = P_k \frac{\partial q}{\partial t}$$

and if $q|_{\partial D \times (0,\infty)} \equiv 0$ then

$$\Delta(P_k q) = P_k \Delta q.$$

Since $\partial_t w_k - \partial_t w = u^m - P_k u^m$ and $H_{2^{-k}}$ is an approximation of the unity it follows that

$$\iint_{D\times(a,b)} \left| \frac{\partial w_k}{\partial t} - \frac{\partial w}{\partial t} \right| dxdt \to 0 \quad \text{as } k \uparrow \infty.$$

Since $P_k(x,t) = \int G_k(x,y)u(y,t)dy$ is a decreasing function of t, and the support of λ is contained in $[1/2, 1]$ it follows that if $t > \tau > 0$ then

$$w_k(x,t) = \int P_k(x, t + 2^{-k}s)\lambda(s)\, ds \leq G_k f_\tau \leq G f_\tau$$

where $f_\tau(x) = \frac{2}{\tau}\int_{\tau/2}^{\tau} u(x,s)\, ds$ (since $p(x, t + s2^{-k}) \leq P_k(x,r)$ for $r \in (\tau/2, \tau)$). □

From the above lemma it follows that the limit $\lim_{k\to\infty} w_k(x,t)$ exists for all $(x,t) \in \Omega$ (it may be $\equiv +\infty$). Hence, from now on we redefine w to be equal to this limit.

Lemma 5.1.4. *For every $\tau > 0$ the trace ν_τ satisfies $d\nu_\tau = w(x,\tau)dx$, and $w(x,\tau) = G\mu_\tau(x)$, where μ_τ is a non-negative measure with finite mass (i.e $\int_D d\mu_\tau < \infty$). Moreover*

$$\lim_{t\downarrow\tau} w(x,t) = w(x,\tau).$$

Proof. Pick $\tau > 0$, $\theta \in C_0^\infty(D)$ and $\gamma \in C_0^\infty(\mathbb{R})$ with $\gamma(\tau) = 1$. Then by definition

$$\int_D \theta(x)\, w_k(x,\tau)\, dx = -\iint_{D\times(\tau,\infty)} \theta(x) \left[\gamma(t)\frac{\partial w_k}{\partial t} + \gamma'(t) w_k\right] dxdt.$$

By Lemma 5.1.3, $\partial_t w_k \to \partial_t w$ in $L^1(D \times \text{supp}(\gamma))$ and $w_k \uparrow w$ as $k \to \infty$ which shows that $d\nu_\tau = w(x,\tau)\, dx$. Therefore $w(x,t)$ is the increasing limit of superharmonic functions $\not\equiv +\infty$ (because $w(x,\tau) \leq G f_\tau$ with $0 \leq f_\tau \in L^1(D)$).

We recall the following well-known result: *A super-harmonic function is a Green's potential in D if and only if the largest minorant is identically zero.*

Since $w(x,\tau)$ is bounded by a potential it follows that

$$w(x,t) = G\mu_\tau$$

for some non-negative measure μ_τ. To see that μ_τ has finite mass we notice that if $h_j = G\sigma_j, \sigma_j \geq 0$, is an increasing sequence of Green's potentials with $\sigma_j(\cdot)$ having compact support in D and $\lim_{j\to\infty} h_j(x) = 1$ for any $x \in D$, then

$$\int h_j d\mu_\tau \equiv \int w(x,\tau)d\sigma_j \leq \int Gf_\tau d\sigma_j = \int h_j f_\tau dx \leq \int f_\tau dx.$$

Letting $j \to \infty$ we obtain that $\int d\mu_\tau < \infty$.

We remark that if $\{D_j\}$ is a sequence of domains such that $\bar{D}_j \subset\subset D_{j+1}$ and $\bigcup D_j = D$, then h_j can be defined such that $h_j = 1$ in D_j, $h_j = 0$ on ∂D, and $\Delta h_j \geq 0$ in D. Also notice that since w_k are decreasing in t so is w. Therefore $h_\tau(x) = \lim_{t\downarrow\tau} w(x,t)$ exists for all $x \in D$ and is superharmonic.

Now pick $\eta \in C_0^\infty(\mathbb{R}^n)$ such that $\eta \geq 0$ and $\int \eta(x)\,dx = 1$. For $\xi \in \mathbb{R}^n$ and $\varepsilon > 0$ define $\eta_{\xi,\varepsilon}(x) = \varepsilon^{-n}\eta((\xi - x)/\varepsilon)$. Thus, for any $\xi \in D$ and any $\varepsilon \leq \operatorname{dist}(\xi,\partial D)/2$, we have

$$\int_D \eta_{\xi,\varepsilon}(x)w(x,\tau)\,dx = \int_D \eta_{\xi,\varepsilon}(x)h_\tau(x)\,dx$$

for all $\xi \in D$, by the weak continuity in t of $w(x,t)$. Taking limit when ε tends to zero we obtain that $w(x,\tau) = h_\tau(x)$ a.e. But both are superharmonic and hence every point is a Lebesgue point. Therefore $w(x,\tau) \equiv h_\tau(x)$, which completes the proof of Lemma 5.1.4 □

Step 3. *Proof of Theorem* 5.1.1. Let $H_0(D) = \operatorname{clos}\left\{\theta \in C_0^\infty(D) : \int_D |\nabla\theta|^2\,dx < \infty\right\}$. If μ is a measure on D we have that

$$G\mu \in H_0(D) \quad \text{if and only if} \quad \mathcal{E}(\mu) = \int_D G(\mu)d\mu < \infty.$$

Moreover

$$\|G\mu\|^2_{H^0(D)} = \mathcal{E}(\mu).$$

Roughly speaking, if $\theta = G\mu$

$$\int_D |\nabla\theta|^2\,dx = -\int \theta\Delta\theta dx = \int G(\mu)d\mu.$$

Lemma 5.1.5. *Let $\mu, \mu_1, \mu_2, \ldots$ be a sequence of non-negative measures in D such that*

(i) $G(\mu_j) \leq G(\mu_{j+1}) \leq G(\mu)$,

(ii) $\mathcal{E}(\mu) < \infty$.

Assume that $\lim_{j\to\infty} G(\mu_j) = G(\mu)$ *for any* $x \in D$. *Then* $\mathcal{E}(\mu_j) \uparrow \mathcal{E}(\mu)$ *and*

$$\|G\mu_j - G\mu\|_{L^2} \to 0$$

as $j \to \infty$.

Proof. By hypothesis

$$\begin{aligned}\mathcal{E}(\mu_j) &= \int G\mu_j\, d\mu_j \le \int G\mu_{j+1}\, d\mu_j \\ &= \int G\mu_j\, d\mu_{j+1} \le \int G\mu_{j+1}\, d\mu_{j+1} = \mathcal{E}(\mu_{j+1}).\end{aligned}$$

Thus $\mathcal{E}(\mu_j) \le \mathcal{E}(\mu_{j+1})$. If $j > k$ then

$$\mathcal{E}(\mu_j) \ge \int G\mu_k\, d\mu_j = \int G\mu_j\, d\mu_k.$$

Thus by the Monotone Convergence Theorem

$$\lim_{j\to\infty} \mathcal{E}(\mu_j) \ge \int G\mu\, d\mu_k = \int G\mu_k\, d\mu.$$

Using again the Monotone Convergence Theorem we obtain that

$$\lim_{j\to\infty} \mathcal{E}(\mu_j) = \mathcal{E}(\mu).$$

To complete the proof of the lemma observe that

$$\|G\mu_j - G\mu\|_{L^2}^2 = \mathcal{E}(\mu_j) + \mathcal{E}(\mu) - 2\int G\mu_j d\mu \to 0$$

as $j \to \infty$. □

Now, let us get back to the proof of Theorem 5.1.1. For $\tau > 0$ we define $u_{k,\tau}^*$ as the strong solution of the IDP (5.1.1) in $D \times (\tau, \infty)$ that satisfies $u_{k,\tau}^* = 0$ on $\partial D \times (\tau, \infty)$ and $u_{k,\tau}^* = u_{k,\tau}$ for $x \in D$ where $u_k(x, \tau) = -\Delta w_k(x, \tau) \ge 0$.

Set $w_{k,\tau}^* = G u_{k,\tau}^*$. Since by previous result w_k^* is decreasing in t it follows that for $(x, t) \in D \times (\tau, \infty)$

$$w_{k,\tau}^*(x, t) \le w_{k,\tau}^*(x, \tau) = w_{k,\tau}(x, \tau) = w_k(x, \tau).$$

We recall that w_k is smooth and solves the inequality $\partial_t w_k \le \psi(\Delta w_k)$ and $\partial_t w_{k,\tau}^* = \psi(\Delta w_{k,\tau}^*)$ (by definition) where $\psi(s) = \operatorname{sgn}(s)\,|s|^m$. Thus by the maximum principle we have that

$$w_k \le w_{k,\tau}^* \quad \text{in } D \times (\tau, \infty). \tag{5.1.11}$$

Letting $\alpha(x, t)$ be the extremal (exceptional) solution of the porous medium equation described in Section 4.2 of Chapter 3 it follows that

$$u_{k,\tau}^*(x, t) \le \alpha(x, t - \tau) \quad \text{for } (x, t) \in D \times (\tau, \infty).$$

Thus the family $\{u^*_{k,\tau}\}$ is uniformly bounded and hence equicontinuous on $D \times [0, \infty)$ for every $a > \tau$. Then a subsequence will converge uniformly to a strong solution u^*_τ of the porous medium equation in $D \times [0, \infty)$ for each $a > \tau > 0$.

Again by the maximum principle, $\{w^*_{k,\tau}\}$ is a monotonically increasing sequence of functions. Therefore the following limit exists

$$\lim_{k\to\infty} w^*_{k,\tau} = G\mu^*_\tau = w^*_\tau.$$

In particular, by (5.1.11) we obtain that

$$w(x,t) \leq w^*_\tau(x,t) \quad \text{for } (x,t) \in D \times (\tau, \infty).$$

Letting $M(t) = \sup\{w(x,t) : x \in D\}$ and observing that the τ above was arbitrary we see that $M(t) < \infty$ for all $t > 0$ and $w^*_\tau(x,t) \leq M(t)$ for $t > \tau$. We claim that

$$\lim_{t\downarrow\tau} \|w(\cdot,t) - w(\cdot,\tau)\|_{H_0(D)} = 0 \quad \text{for all } \tau > 0, \tag{5.1.12}$$

in other words that $w(\cdot,t)$ is continuous in the H_0-norm. Indeed, for all $\tau > 0$ we have

$$\begin{aligned}\lim_{t\downarrow\tau} \|w(\cdot,t) - w(\cdot,\tau)\| &= \lim_{t\downarrow\tau} \|w^*_\tau(\cdot,t) - w(\cdot,\tau)\| \\ &= \lim_{k\to\infty} \|w_k(\cdot,\tau) - w(\cdot,\tau)\| = 0\end{aligned}$$

which proves the claim.

Now let $e_k(t) = \|w_k(\cdot,t)\|^2_{H_0}$. Since w_k is smooth we have

$$\frac{de_k(t)}{dt} = -\frac{d}{dt}\int_D w_k(x,t)\,\Delta w_k(x,t)\,dx = -2\int_D \frac{\partial w_k}{\partial t}\,\Delta w_k\,dx.$$

But $\Delta w_k = -P_k u$ and $\partial_t w_k = -P_k u^m$, where P_k is the approximation of the identity defined in (5.1.10). Thus for $0 < \tau < T$, we have

$$2\iint_{D\times(\tau,T)} P_k u\, P_k u^m\,dx \equiv e_k(\tau) - e_k(T) \leq C(\tau,T).$$

Since $P_k u$ and $P_k u^m$ converge pointwise a.e. to u and u^m respectively, by Fatou's Lemma we obtain that

$$\iint_{D\times(\tau,T)} u^{m+1}\,dxdt < \infty$$

and hence

$$\iint_{D\times(\tau,T)} \left(|P_k u - u|^{m+1} + \left|P_k u^m - u^m\right|^{\frac{m+1}{m}}\right)dxdt \to 0$$

as $k \to \infty$. Differentiating the energy of $w_k(\cdot,t) - w_\tau^*(\cdot,t)$ we see that for $0 < \tau < T$ that

$$\begin{aligned}\|w_k(\cdot,\tau) - w_\tau^*(\cdot,\tau)\|^2 \\ &= \|w_k(\cdot,T) - w_\tau^*(\cdot,T)\|^2 + \iint_{D\times(\tau,T)} \left(P_k u - u_\tau^*\right)\left(P_k u^m - (u_\tau^*)^m\right) dxdt.\end{aligned}$$

Letting $k \to \infty$ we obtain that

$$\|w(\cdot,T) - w_\tau^*(\cdot,T)\|^2 + \iint_{D\times(\tau,T)} (u - u_\tau^*)\left(u^m - (u_\tau^*)^m\right) dxdt = 0.$$

Hence $u = u_\tau^*$ a.e. in $D \times (\tau, \infty)$ which yields Theorem 5.1.1. □

5.2 Weak solutions of the porous medium equation

In this section we shall study the local regularity of weak solutions of the porous medium equation. More precisely, let $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}$ be an open set and let $u \in L^m_{\text{loc}}(\Omega)$ with $m > 1$ and $n \geq 0$, be a weak solution of the porous medium equation $\partial u/\partial t = \Delta u^m$, $m > 1$, in Ω, i.e.

$$\frac{\partial u}{\partial t} = \Delta u^m \quad \text{in } D'(\Omega) \text{ (i.e. in the distribution sense).} \tag{5.2.1}$$

Our main goal in this section is to show the following regularity result [47].

Theorem 5.2.1. *Suppose $u \geq 0$ is a weak solution of the porous medium equation in the domain Ω. Then there exists a continuous weak solution of u^* of (5.2.1) such that $u = u^*$ a.e. in Ω.*

Remark. As it was pointed out in the introduction, u need not be more smooth than Hölder continuous. Furthermore by the result of E. DiBenedetto and A. Friedman [67] each continuous solution is Hölder continuous.

We divide the proof in three steps. First we prove that if $\Omega_0 = D \times (a,b) \subset\subset \Omega$ with ∂D smooth then u has a trace on $\partial_p \Omega_0$ (parabolic boundary of Ω_0). Next we find u^* which is a continuous weak solution of the porous medium equation in Ω_0 with the same trace as u. Finally we show that $u = u^*$ a.e. in Ω_0. For the convenience of the reader we shall first outline the proofs of the three steps and in the sequel we shall provide the details.

Step 1. *Existence of trace on the parabolic boundary of* Ω_0. First we consider the bottom of Ω_0. We shall prove that for each $\tau \in I$ there exists a unique finite non-negative measure v_τ on D with

$$\sup_{\tau\in I} \int_D dv_\tau < \infty$$

such that v_τ is the trace of u at $t = \tau$ in the sense that

$$\iint_{\Omega_0 \cap \{t>\tau\}} \left(u^m \, \Delta\eta + u \, \frac{\partial \eta}{\partial t} \right) dx dt = - \int_D \eta(x, \tau) \, dv_\tau$$

for any $\eta \in A \equiv C_0^\infty(\Omega_0)$.

To establish this result we first show that the expression

$$\iint_{\Omega_0 \cap \{t>\tau\}} \left(u^m \Delta\eta + u \frac{\partial \eta}{\partial t} \right) dx dt$$

depends only of the values of η at $t = \tau$ and is non-positive. Also we shall prove that v_τ is weakly continuous in τ.

Next we find the trace on the lateral side of Ω_0. Thus we shall have that there exists a unique non-negative measure μ on $S = \partial D \times (a, b)$ such that

$$\iint_{\Omega_0} \left(u^m \Delta\eta + u \frac{\partial \eta}{\partial t} \right) dx dt = - \iint_S \frac{\partial \eta}{\partial t} \, d\mu$$

for all $\eta \in \mathscr{B} = C_0^\infty\big(\Omega \cap \{(x, t) : \ a < t < b\}\big)$ such that $\eta(x, t) = 0$ whenever $x \in \partial D$. Moreover μ is weakly continuous with respect to ∂D.

Step 2. *A priori estimate for continuous weak solutions.* Let $u^* \geq 0$ be a solution in $C(\overline{D} \times [0, T])$ of

$$\begin{aligned} \partial_t u &= \Delta u^m && \text{in } D \times (0, T), \\ u(x, 0) &= f(x) && x \in D, \\ u^m &= g && \text{on } \partial D \times [0, T]. \end{aligned}$$

We shall prove the following a priori estimate for u^*. If

$$\int \delta(x) \, f(x) \, dx < L \quad (\delta(x) = \operatorname{dist}(x, \partial D))$$

and

$$\int_{\partial D} \int_0^T g \, dt d\sigma_x < L$$

then

$$\iint_\Omega (u^*)^m \, dx dt \leq C(L)$$

and

$$\sup_{t \in I} \int_D \delta(x) \, u^*(x, t) \, dx \leq C(L).$$

Step 3. *Equality* $u = u^*$ *a.e. in* Ω_0. Here we shall use Potential Theory. Define

$$w(x, t) = \int_D G(x, y) \, dv_t$$

i.e., the Green's potential of v_t, and introduce the harmonic function

$$h(x,t) = \int_{\partial D} \int_a^t \frac{\partial G}{\partial \eta_\xi}(\xi, X)\, d\mu(\xi)$$

with

$$\frac{\partial G}{\partial \eta_\xi}(\xi, X)|_{\xi \in \partial D} \equiv \text{Poisson's kernel.}$$

Notice that

(i) $\Delta h = 0$

(ii) $\int_D h(x,t)\, dx \le C$ by Fubini's theorem

(iii) $h \ge 0$.

Thus we can use Harnack's principle to show that

$$\sup_{K \times [a,b]} h < \infty \quad \text{for } K \subset\subset D.$$

Now define $q = w - h$ so that

$$\frac{\partial q}{\partial t} + u^m = 0 \tag{5.2.2}$$

and

$$\Delta q = -u$$

in the distribution sense. Also by adding a constant we can assume that in a compact set $q > 0$, i.e., there exists $M = M(K)$ such that $Q = q + M > 0$ in $K \times [a,b]$.

Now using Moser iteration it follows that Q is bounded in $K \times [a,b]$. Therefore w is bounded, since $w - h \equiv q$.

Also by integrating (5.2.2) it follows that $\int_a^t u^m(x,s)\, ds$ is bounded in Ω_0. Hence

$$w(x,t) = Gv_a(x) - \int_a^t u^m(x,s)\, ds + h(x,t).$$

Using the above identity we shall show that

$$u \in L^{m+1}_{\text{loc}}(\Omega) \tag{5.2.3}$$

using the following lemma.

Lemma 5.2.2. *Let w be a smooth function defined in a neighborhood of Ω_0 such that*

(i) $\partial w / \partial t \le 0$,

(ii) $\Delta w \le 0$,

(iii) $\int_D |\Delta w(x,t)|\, dx < M$ *for each* $t \in [a,b]$,

(iv) $|w| \leq M$.

Then for any $K \subset\subset D$ *there exists* $C = C(K, \Omega_0)$ *such that*

$$\iint_{K\times(a,b)} \frac{\partial w}{\partial t}\,\Delta w\,dxdt < C\,M^2.$$

Applying this lemma to $Gv_a(x) - \int_a^t u^m(x,s)ds$ we obtain (5.2.3). Once the (5.2.3) has been established we use energy methods (as in the previous section) to complete the proof of the theorem. We proceed now to the detailed proof of the theorem.

Proof of Theorem 5.2.1. *Step* 1. We have $u \in L^m_{\text{loc}}(\Omega)$, $u \geq 0$ such that $\partial_t u = \Delta u^m$ in $D'(\Omega)$. Let $\Omega_0 = D \times [a,b] \subset\subset \Omega$ with ∂D smooth.

Lemma 5.2.3. *For any* $\tau \in I = (a,b)$ *there exists a unique non-negative measure* v_τ *on D such that*

$$\iint_{\Omega_0\cap\{t>\tau\}} \left(u^m \Delta\eta + u\frac{\partial\eta}{\partial t}\right) dxdt + \int_D \eta(x,\tau)\,dv_\tau = 0$$

for any $\eta \in A = C_0^\infty(\Omega_0)$ *with*

$$\sup_{\tau\in I}\int_D dv_\tau < \infty.$$

Proof. For each $\tau \in I$ and each $\eta \in A$ we define the functional

$$\Lambda_\tau(\eta) = \iint_{\Omega_0\times\{t>\tau\}} \left(u^m\,\Delta\eta + \frac{\partial\eta}{\partial t}\right) dxdt.$$

Claim. If $\eta_1, \eta_2 \in A$ with $\eta_1 \equiv \eta_2$ at $t \equiv \tau$ then $\Lambda_t(\eta_1) = \Lambda_t(\eta_2)$.

Indeed, let

$$\eta(x,t) = \begin{cases} \eta_1 - \eta_2 & \text{for } t > \tau, \\ 0 & \text{otherwise.} \end{cases}$$

Pick $\psi \in C_0^\infty(\mathbb{R})$ with $\int\psi = 1$ and $\operatorname{supp}\psi \subseteq \mathbb{R}^+$. Define $\eta_\varepsilon(x,t) = \int \eta(x, t - \varepsilon s)\,\psi(s)\,ds$. Notice that $\eta_\varepsilon(x,t) = 0$ for $t \leq \tau$, $\eta_\varepsilon \in A$, and $\Lambda_t(\eta_\varepsilon) \equiv 0$. Since $\partial\eta/\partial t \in L^\infty$ and $\partial\eta_\varepsilon/\partial t \to \partial\eta/\partial t$ a.e. we obtain that $\Lambda_t(\eta) = 0$ which proves the claim.

Now for $\gamma \in C_0^\infty(D)$ we define the distribution $\lambda_\tau(\,\cdot\,)$ as $\lambda_\tau(\gamma) = \Lambda_\tau(\eta)$ for $\eta \in A$ such that $\eta(x,\tau) = \gamma(x)$.

Claim. The distribution λ_τ is given by a non-positive measure.

Indeed, pick $\psi \in C_0^\infty(\mathbb{R})$ with $\psi(0) = 1$ and $\psi'(s) \geq 0$ for $s \leq 0$. For $\varepsilon > 0$ define $\psi_\varepsilon(s) = \psi(s/\varepsilon)$ and $\eta_\varepsilon(x,t) = \psi_\varepsilon(x - \tau)\gamma(x)$. Thus by definition of $\Lambda_\tau(\,\cdot\,)$

and the weak solution

$$\lambda_\tau(\gamma) = \iint_{\Omega_0 \times \{t \le \tau\}} \left(u^m \, \Delta\gamma(x) \, \psi_\varepsilon(t-\tau) + u \, \gamma(x) \, \frac{\partial \psi_\varepsilon}{\partial t}(t-\tau) \right) dxdt$$
$$\le - \iint_{\Omega_0 \times \{t \le \tau\}} u^m \, \Delta\gamma(x) \, \psi_\varepsilon(t-\tau) \, dxdt \to 0 \quad \text{as } \varepsilon \downarrow 0$$

which proves the claim and finishes the proof of the lemma. □

Next we are concerned with the lateral side, as our next lemma shows.

Lemma 5.2.4. *Let I and D be defined as above. Then there exists a unique bounded and non-negative measure μ on $S = \partial D \times I$ such that*

$$\iint_{\Omega_0} \left(u^m \Delta\eta + u \frac{\partial \eta}{\partial t} \right) dxdt + \iint_S \frac{\partial \eta}{\partial n} \, d\mu = 0$$

for any $\eta \in B := C_0^\infty\big(\Omega \cap \{(x,t) : a < t < b\}\big)$ such that $\eta(x,t) \equiv 0$ whenever $x \in \partial D$.

Proof. For $\eta \in \beta$ we define the functional

$$\Lambda(\eta) = \iint_{\Omega_0} \left(u^m \Delta\eta + u \frac{\partial \eta}{\partial t} \right) dxdt.$$

Claim. If $\eta \in B$ and $\frac{\partial \eta}{\partial n}|_S = 0$ then $\Lambda(\eta) \equiv 0$.

Indeed, pick $\psi \in C_0^\infty(\mathbb{R}^n)$, with $\psi \ge 0$ and $\int \psi = 1$. For each $\eta \in B$ define

$$\eta_\varepsilon(x,t) = \varepsilon^{-n} \int_D \eta(\xi,t) \psi \left(\frac{x-\xi}{\varepsilon} \right) d\xi.$$

Notice that $\eta_\varepsilon \in C_0^\infty(\Omega)$ for ε sufficiently small. Thus,

$$\Delta\eta_\varepsilon(x,t) = \varepsilon^{-n} \int_D \Delta\eta(\xi,t) \, \psi \left(\frac{x-\xi}{\varepsilon} \right) d\xi$$

and since u is a weak solution

$$\Lambda(\eta) = \lim_{\varepsilon \to 0} \iint_{\Omega_0} \left(u^m \Delta\eta_\varepsilon + u \, \frac{\partial \eta_\varepsilon}{\partial t} \right) dxdt = 0$$

which proves the claim.

Now for $\gamma \in C_0^\infty(S)$ $(S = \partial D \times (a,b) = \partial D \times I)$ we define

$$\lambda(\gamma) = \Lambda(\eta)$$

by picking an $\eta \in B$ such that $\frac{\partial \eta}{\partial n} = \gamma$. By the above claim $\lambda(\,\cdot\,)$ is well defined.

Claim. $\lambda(\gamma) \leq 0$ whenever $\gamma \geq 0$.
Indeed, for $\eta \in B$ define

$$\tilde{\eta} = \begin{cases} \eta & \text{in } \Omega_0, \\ 0 & \text{otherwise,} \end{cases}$$

so that if $\gamma = \frac{\partial \eta}{\partial n} \geq 0$ then for any $\theta \in C_0^\infty(D)$, with $\theta \geq 0$

$$\begin{aligned} \int_D \Delta\tilde{\eta}\,\theta &= \int_D \tilde{\eta}\Delta\theta = \int_D \eta\,\Delta\theta \\ &= -\int_{\partial D} \eta\,\frac{\partial\theta}{\partial n} + \int_{\partial D} \frac{\partial\eta}{\partial n}\,\theta + \int_D \Delta\eta\,\theta \geq \int_D \Delta\eta\,\theta. \end{aligned}$$

Therefore

$$\Delta\tilde{\eta} \geq \chi_{\Omega_0}\Delta\eta.$$

Consequently

$$\begin{aligned} \lambda(\gamma) = \iint_{\Omega_0} \left(u^m \Delta\eta_\varepsilon + u\,\frac{\partial\eta_\varepsilon}{\partial t} \right) dxdt &\leq \iint_{\Omega_0} \left(u^m \Delta\tilde{\eta} + u\,\frac{\partial\tilde{\eta}}{\partial t} \right) dxdt \\ &= \lim_{\varepsilon\to 0} \iint_{\Omega_0} \left(u^m \Delta\tilde{\eta}_\varepsilon + u\,\frac{\partial\tilde{\eta}_\varepsilon}{\partial t} \right) dxdt = 0 \end{aligned}$$

which proves the claim. □

Lemma 5.2.5. *Let* $\psi \in C(\Omega_0)$ *such that* $\operatorname{supp}\psi \subset\subset \Omega_0$. *Then*

$$\int_D \psi(x,\tau)dv_\tau$$

is a continuous function of τ *and*

$$\int_a^b \left(\int_D \psi(x,\tau)\,dv_\tau \right) d\tau = \iint_{\Omega_0} u\,\psi\,dxd\tau.$$

Proof. By an approximation argument it suffices to prove the lemma for $\psi \in C_0^\infty(\Omega_0)$. By definition we have that

$$\int_D \psi(x,\tau)dv_\tau = \int_{\Omega\cap\{t>\tau\}} \left(u^m \Delta\psi + u\,\frac{\partial\psi}{\partial t} \right) dxdt.$$

Thus the continuity follows immediately. Also

$$\int_a^b \left(\int_D \psi(x,\tau)dv_\tau \right) d\tau = \int \left\{ \iint_{\Omega\cap\{t>\tau\}} \left(u^m \Delta\psi + u\,\frac{\partial\psi}{\partial t} \right) dxdt \right\} d\tau.$$

By defining $q(x,t) \equiv (t-a)\psi(x,t)$ and changing the order of integration the above expression is equal to

$$\iint u\,\psi\,dxdt - \iint \left(u^m \Delta q + u\,\frac{\partial q}{\partial t} \right) dxdt = \iint u\,\psi\,dxd\tau$$

which completes the proof of Lemma 5.2.5. □

We shall now give our most general trace result.

Lemma 5.2.6. *Let μ, υ_τ be defined as above. Let $\psi \in C_0^\infty(\Omega)$ such that $\psi \equiv 0$ on $S = \partial D \times [a, b]$. Then*

$$\int_D \psi(x, b)\, d\upsilon_b = \int_D \psi(x, a) d\upsilon_a + \iint_{\Omega_0} \left(u^m \Delta\psi + u \frac{\partial \psi}{\partial t} \right) dx dt + \iint_S \frac{\partial \psi}{\partial n}\, d\mu. \tag{5.2.4}$$

Proof. Pick $\eta_k \in C_0^\infty(\mathbb{R})$ with $0 \le \eta_k \le 1$ and with support in (a, b) such that $\eta_k' dt \to d\delta_a - d\delta_b$ weakly (where δ_c is the Dirac measure at c). We may also assume that

$$\lim_{k\to\infty} \eta_k = \begin{cases} 1 & t \in [a, b], \\ 0 & \text{otherwise.} \end{cases}$$

Using $q = \eta_k \psi$ as a test function in the class B defined previously, it follows from Lemma 5.2.4 that

$$\iint_{\Omega_0} \left(u^m\, \eta_k\, \Delta\psi + u\, \eta_k \frac{\partial \psi}{\partial t} \right) dx dt + \iint_S \eta_k \frac{\partial \psi}{\partial n}\, d\mu = - \iint_{\Omega_0} u\, \eta_k'\, \psi\, dx dt.$$

Letting $k \to \infty$ and using the continuity result in Lemma 5.2.5 we obtain (5.2.4). □

Step 2. As before $D \subseteq \mathbb{R}^n$ open bounded set with ∂D smooth $I = (0, T)$ and $D \times I = \Omega_0 \subset\subset \Omega$. Let $u \in L^m_{\text{loc}}(\Omega),\ u \ge 0$ be a solution of the IDP

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u^m && \text{in } D'(\Omega), \\ u(x, 0) &= f(x) && x \in D, \\ u^m &= g && \text{on } \partial D \times [0, T]. \end{aligned}\right\} \tag{5.2.5}$$

Proposition 5.2.7. *Assume that $u \in C(\bar{\Omega})$ solve (5.2.5), and that*

$$\int_D \delta(x)\, f(x)\, dx < L \quad (\delta(x) = \operatorname{dist}(x, \partial D))$$

and

$$\iint_S g(x)\, d\sigma_x\, dt < L \quad (S = \partial D \times (0, T)).$$

Then

$$\iint_\Omega u^m dx dt < C(L) \tag{5.2.6}$$

and

$$\sup_{t\in I} \int_D \delta(x)\, u(x, t)\, dx < C(L). \tag{5.2.7}$$

Proof. Define η as the Green's potential of -1, i.e.

$$\begin{aligned} \Delta\eta &= -1 \quad \text{in } D, \\ \eta &= 0 \quad \text{on } \partial D. \end{aligned}$$

We may recall that by properties of the Green's function it follows that

$$C_1\,\delta(x) \le \eta(x) \le C_2\,\delta(x) \quad \text{for all } x \in D.$$

By the Green's formula

$$\begin{aligned} \frac{d}{dt}\int_D u(x,t)\,\eta(x)\,dx &= \int_D \Delta u^m\,\eta\,dx \\ &= -\int_D u^m\,dx + \int_{\partial D} (u(x,t))^m\,\frac{\partial\eta}{\partial n}\,d\sigma(x). \end{aligned} \tag{5.2.8}$$

Hence for $t \in I$

$$\int_D \eta(x)\,u(x,t)\,dx \le \int_D \eta(x)\,f(x)\,dx + \int_{\partial D}\int_0^t g\,\frac{\partial\eta}{\partial n}\,d\sigma\,dt$$

which proves (5.2.7). Using (5.2.7), the estimate (5.2.6) follows by integrating (5.2.8) in t from 0 to T. □

Step 3. We begin by proving Lemma 5.2.2 stated above.

Proof of Lemma 5.2.2. Without loss of generality we can assume that $w \ge 0$ and $M = 1$.

First we prove the lemma for $w \in C^\infty(\bar{\Omega}_0)$ such that $w|_{\partial D\times(a,b)} = 0$. Thus

$$\frac{d}{dt}\int_D w\,\Delta w = \int_D \frac{\partial w}{\partial t}\,\Delta w + \int_D w\,\frac{\partial \Delta w}{\partial t} = 2\int_D \frac{\partial w}{\partial t}\,\Delta w$$

and

$$\begin{aligned} &\int_D w(x,b)\,\Delta w(x,b)\,dx - \int_D w(x,a)\,\Delta w(x,a)\,dx \\ &\quad = 2\int_a^b\int_D \frac{\partial w}{\partial t}\,\Delta w \le C \end{aligned}$$

which proves the desired result. □

Now we consider a general $w \in C^\infty(\Omega_0)$ which satisfies the hypothesis of our lemma. We claim that there exists $p > 1$ such that:

$$\text{if } \tilde{K} \subset\subset D \text{ then } \left(\int_{\tilde{K}} |\nabla w|^p\,dx\right)^{1/p} \le C. \tag{5.2.9}$$

To see the claim, let $\tilde{K} \subset\subset \tilde{D} \subset\subset D$ with $\partial\tilde{D}$ smooth. We can write w restricted to $\tilde{D}$ in the following form:

$$w = \text{Poisson's integral of } w|_{\partial\tilde{D}} + \text{Green's potential of } \Delta w_{\tilde{D}} \equiv u + v.$$

Since u is harmonic we have that

$$\|\nabla u\|_{L^\infty(\tilde{K})} \le C(\tilde{K}) \; \|u\|_{L^\infty(\tilde{D})} \le C_1(\tilde{K}).$$

Also using that

$$\left|\nabla_x G_{\tilde{D}}(x, y)\right| \le \frac{C}{|x-y|^{n-1}}$$

it follows that

$$|\nabla v(x)| \le C \int_{\tilde{D}} |\Delta w(y)| \; \frac{1}{|x-y|^{n-1}} \, dy.$$

But the operator $L\colon L^1(\tilde{D}) \to L^{p_0}_{\text{weak}}(\mathbb{R}^n)$ defined as

$$f \to f * \frac{1}{|x|^{n-1}}$$

is continuous for $p_0 = \frac{n}{n-1}$. Therefore taking $p < p_0$, and using the compactness of the domain we complete the proof of the claim.

Now going back to the proof of the lemma, we define D_τ, $\tau \in [0, 1]$, a one parameter family of domains such that

(i) $K \subset\subset D_\tau$ for all $\tau \in [0, 1]$,

(ii) ∂D_τ is smooth,

(iii) $D_\tau \subset\subset D_{\tau'}$ if $\tau < \tau'$,

(iv) $D_1 = \tilde{D}$.

For each $\tau \in [0, 1]$ we introduce the function w_τ where $w_\tau = w$ in D_τ and w_τ harmonic in $D - D_\tau$ with boundary values 0 on $\partial\tilde{D}$ and w on ∂D_τ. Thus

(i) $0 \le w_\tau \le 1$ (maximum principle),

(ii) $\frac{\partial w_\tau}{\partial t} \le 0$ (hypothesis),

(iii) w_τ is continuous on $\tilde{D}$ and in τ on [0,1],

(iv) w_τ is superharmonic (note that $w_\tau \le w$).

Since $w_\tau|_{\partial\tilde{D}} = 0$ we can apply the result proven above if we prove the estimate

$$\sup_{(a,b)} \int_{\tilde{D}} |\Delta w_\tau(x, t)| \, dx < M.$$

But from the above estimate

$$\iint_{K\times(a,b)} \frac{\partial w}{\partial t}\,\Delta w\,dxdt \le \iint_{\tilde{D}\times(a,b)} \frac{\partial w_\tau}{\partial t}\,\Delta w_\tau\,dx\,dt \le C\int_D |\Delta w_\tau(x,a)|\,dx.$$

Therefore

$$\iint_{K\times(a,b)} \frac{\partial w}{\partial t}\,\Delta w\,dxdt \le C\int_{1/2}^{3/4}\int_D |\Delta w_\tau(x,a)|\,dx\,dt.$$

Let us estimate Δw_τ. Since

$$\Delta w_\tau = (\Delta w)\,\chi_{D_\tau} + \left(\frac{\partial w}{\partial n} - \frac{\partial w_\tau}{\partial n}\right) d\sigma_{\partial D_\tau}$$

we only have to bound

$$\begin{aligned}
&\int_{1/2}^{3/4}\int_{\partial D_\tau}\left(\left|\frac{\partial w}{\partial n}\right| + \left|\frac{\partial w_\tau}{\partial n}\right|\right) d\sigma_{\partial D_\tau}\,d\tau \\
&\qquad \le C\int_{D_{3/4}\setminus D_{1/2}} |\nabla w|\,dx + C\int_{1/2}^{3/4}\left(\int_{\partial D_\tau}\left|\frac{\partial w_\tau}{\partial n}\right|^p d\sigma\right)^{1/p} d\tau \\
&\qquad = \mathrm{I} + \mathrm{II}.
\end{aligned}$$

The integral I can be estimated by using (5.2.9). For II we combine the regularity result that allows us to estimate the norm of the normal derivative in terms of tangential derivatives together with Hölder's inequality to obtain

$$\mathrm{II} \le \int_{1/2}^{3/4}\left(\int_{\partial D_\tau} |\nabla_T w_\tau|^p\,dx\right)^{1/p} d\tau \le C\left(\iint_{D_{3/4}\setminus D_{1/2}} |\nabla w|^p\right)^{1/p} \le C$$

by (5.2.9). □

Next we shall show that $w(x,t) = \int_D G(x,y)\,dv_t$ (where v_t is the trace of u given by Lemma 5.2.3) is locally bounded. For this purpose we shall use a variant of the Moser [109] iteration technique introduced in [45]. We begin by establishing a useful consequence of the classical Sobolev inequality.

Lemma 5.2.8. *Let F be a non-negative smooth function in $\Omega_0 = D\times(a,b)$. Suppose that $m > 1$ and $K \subset\subset D$ are given. Then there exist constants $k > 1$ and $\theta > 0$ such that if $H \ge 0$ and $\Delta F \ge -H$ then*

$$\begin{aligned}
\iint_{K\times[a,b]} F^{km} \le C\Bigg\{&\iint_{\Omega_0} H^m\,dxdt\cdot\sup_{t\in[a,b]}\left(\int_D F(x,t)\,dx\right)^{m(k-1)} \\
&+ \operatorname{dist}(K,\partial D)^{-\theta}\cdot\sup_{t\in[a,b]}\left(\int_D F(x,t)dx\right)^{mk}\Bigg\}.
\end{aligned}$$

Proof. Let

$$p = \begin{cases} \frac{n}{n-2m} & \text{if } n > 2m, \\ 2 & \text{otherwise,} \end{cases}$$

and set $q = \frac{p}{p-1}$. For $k > 1$ and $t \in [a, b]$ it follows, from Hölder's inequality, that

$$\int_K F^{km}(x,t)dx \le \left(\int_K F^{pm}(x,t)\,dx \right)^{1/p} \left(\int_K F^{(k-1)qm}(x,t)\,dx \right)^{1/q}.$$

Next, we choose k such that $(k-1)qm = 1$. Let $GH(x,t)$ be the Green's potential of H, i.e.

$$GH(x,t) = \int_D G(x,y)\,H(y,t)\,dy$$

and define $f = \max(F - GH, 0)$. Then

(i) $f \ge 0$,

(ii) $f \le F$,

(iii) f is superharmonic ($F - GH$, and 0 are superharmonic),

(iv) $F \le f + GH$.

By fractional integration

$$\left(\int_K (GH)^{pm}(x,t)\,dx \right)^{1/p} \le C \int_D H^m(x,t)\,dx.$$

Since f is subharmonic in x

$$f(\xi) \le C\,\mathrm{dist}(K, \partial D)^{-n} \int_D f\,dx.$$

Therefore

$$\left(\int_K f^{pm} \right)^{1/p} \le C\,(\mathrm{dist}\,K, \partial D)^{-nm} \left(\int_D F \right)^m.$$

Adding these estimates and integrating in t completes the proof of the proposition. □

We recall that $w(x,t) = G_D u(x,t)$. Thus by the results proved in Step 1, for any $\eta \in C_0^\infty(\Omega_0)$ we have

$$\begin{aligned} \iint_{\Omega_0} \frac{\partial \eta}{\partial t}\, w\,dxdt &= \int_{\Omega_0} \frac{\partial (G\eta)}{\partial t}\, u\,dxdt \\ &= -\iint_{\Omega_0} \eta\, u^m\,dxdt + \iint_{\partial D \times [a,b]} \frac{\partial (G\eta)}{\partial n} d\mu. \end{aligned}$$

which can be written as

$$\iint_{\Omega_0} \left(\frac{\partial \eta}{\partial t} w + \eta u^m \right) dxdt = \iint_{\partial D \times [a,b]} \frac{\partial (G\eta)}{\partial n} d\mu.$$

Defining the harmonic function h as

$$h(x,t) = \iint_{\partial D \times [a,b]} \frac{\partial G}{\partial n}(\xi, x)\, d\mu(\xi)$$

we have

(i) $\Delta h = 0$,

(ii) $\int_D h(x,t)dx \leq C$,

(iii) $h \geq 0$.

Therefore using Harnack's inequality for harmonic functions, we have the pointwise bound

$$h \leq C(K) \quad \text{for any } K \subset\subset D.$$

Define now $q \equiv w - h$ so that $\Delta q = -u$ in $D'(D)$ and $\partial q/\partial t + u^m = 0$ in $D'(\Omega_0)$. Also, by the above it follows that $\sup_{K \times [a,b]} h < \infty$.

Next, $x_0 \in D$ and set $B_\rho = \{x : |x - x_0| < \rho\}$. Assume that $B_\rho \subset\subset D$, $a < T < \tau < b$, and set

$$S = B_\rho \times (\tau, b], \quad R = B_\rho \times (T, b].$$

Fix M so large such that

$$Q = q + M \geq 0 \quad \text{in } B_\rho \times [a,b]. \tag{5.2.10}$$

Lemma 5.2.9. *Suppose P is a smooth and convex function with a bounded derivative $P' = p \in L^\infty([0,\infty))$. Also suppose that $P(0) = p(0) = 0$. Define Z by*

$$Z(t) = \int_0^t (p(s))^{1/s}\, ds.$$

Then

$$\sup_{s \in [\tau,b]} \int_{B_\rho} P(Q(x,s))\, dx + \iint_S [\max(-\Delta Z(Q), 0)]^m\, dxdt \leq \frac{C}{(\tau - T)} \iint_R P(Q) dxdt.$$

Proof. From the hypothesis we have that P and p are non-negative and non-decreasing.

Fix $0 < r < \rho$ and pick $\eta \in C^\infty(\mathbb{R}^{n+1})$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $B_r \times [\tau, \infty)$ and $\eta \equiv 0$ if $x \in \mathbb{R}^n \setminus B_\rho$ or $t < T$. We may also choose η so that

$$\left| \frac{\partial \eta}{\partial t} \right| \leq \frac{C}{(\tau - T)}$$

where $C = C(n)$.

For $s \in [\tau, b]$ set $S' = B_r \times (\tau, s]$, $R' = B_\rho \times (T, s]$. Since $\partial Q/\partial t + u^m = 0$ we have by multiplying by $P(Q)\eta^2$ and integrating over R'

$$\iint_{R'} p(Q)\,\eta^2 \frac{\partial Q}{\partial t}\,dxdt + \iint_{R'} p(Q)\,\eta^2\,u^m\,dxdt = 0.$$

Integration by parts shows that

$$\int_{B_\rho} P(Q(x,s))\eta^2\,dx + \iint_{R'} p(Q)\,\eta^2\,u^m\,dxdt = 2\iint_{R'} P(Q)\eta\frac{\partial\eta}{\partial t}\,dxdt$$

and hence we can conclude that

$$\sup_{s\in[\tau,b]}\int_{B_\rho} P(Q(x,s))ds + \iint_S p(Q)u^m\,dxdt \\ \leq \frac{C}{(\tau - T)}\iint_R P(Q)\,dxdt.$$

Assuming that $\Delta Z(Q) \geq -(p(Q))^{1/m}u$, we finish the proof of the lemma. To prove that $\Delta Z(Q) \geq -(p(Q))^{1/m}u$, we compute

$$\begin{aligned}\Delta Z(Q) &= \operatorname{div}(\nabla Z(Q)) = \operatorname{div}((p(Q))^{1/m}\cdot\nabla Q)\\ &= (p(Q))^{1/m}\Delta Q + \frac{1}{m}(P(Q))^{1/m-1}\,p'(Q)\,|\nabla Q|^2.\end{aligned}$$

Since $\Delta Q = -u$, and the last term above is non-negative the proof is completed. □

As a consequence we have:

Lemma 5.2.10. *Let $\alpha \geq 0$ and set $\beta = (\alpha + m)/m$. Then*

$$(\alpha+1)^{m-1}\sup_{\tau<s<b}\int_{B_\rho} Q^{\alpha+1}(x,s)\,ds + \iint_S \big(\max(-\Delta Q^\beta, 0)\big)^m\,dxdt \\ \leq \frac{C(\alpha+1)^{m-1}}{\tau - T}\iint_R Q^{\alpha+1}\,dxdt.$$

Proof. For $L > 0$ let p be defined by

$$p(s) = \begin{cases}(\alpha+1)s^\alpha & 0 \leq s \leq L,\\ (\alpha+1)L^\alpha & \text{otherwise.}\end{cases}$$

Using Lemma 5.2.9 with $P(s) = \int_0^s p(t)\,dt$ and then letting $L \to \infty$ yields the lemma. □

We can now prove the boundedness of w.

Lemma 5.2.11. *Let $T \in (a, b)$ and $\rho > 0$ such that*

$$B_{3\rho} = \{x : |x - x_0| < 3\rho\} \subseteq D$$

with $D \times (a, b) = \Omega_0 \subset\subset \Omega$. Then $w = Gu$ is bounded in $R = B_\rho \times [T, b]$.

Proof. Choose M so large in (5.2.10) such that $Q \geq 1$ in $B_{3\rho} \times [a, b]$. For $j = 0, 1, \ldots$ define $\alpha_0 = 0$ and $\alpha_{j+1} = k(\alpha_j + m) - 1$ where k is defined in Lemma 5.2.8. Set $\beta_j = \frac{\alpha_j + m}{m}$, $\rho_j = p\frac{j+2}{j+1}$, $T_j = \frac{a + jT}{1+j}$, $B_j = B_{\rho_j}$, $I_j = (T_j, b]$, $R_j = B_j \times I_j$ and

$$A_j = \left(\iint_{R_j} Q_j^{\alpha_j + 1}\, dxdt \right)^{1/(\alpha_j + 1)}.$$

Since $u \in L^m_{\text{loc}}(\Omega)$ we have that $A_0 < \infty$. Setting

$$E_j = \sup_{s \in I_j} \int_{B_j} Q^{\alpha_j + 1}$$

we have by combining lemmas 5.2.8 and 5.2.10 that

$$\begin{aligned} A_{j+1}^{\alpha_{j+1}+1} &\equiv \iint_{R_{j+1}} Q^{\alpha_{j+1}+1} = \iint_{R_{j+1}} Q^{k\beta_j m} \\ &\leq C \left\{ (\alpha_j + 1)^{m-1} \cdot A_j^{\alpha_{j+1}+1} \cdot E_j^{(\alpha_j + m)(k-1)/(\alpha_j+1)} \cdot (j+1)^2 \right. \\ &\qquad \left. + (j+1)^\theta \cdot E_j^{(\alpha_j + m)k/(\alpha_j + 1)} \right\}. \end{aligned}$$

Observe that $(j + 1)^\theta \leq C\,(\alpha_j + 1)^{m-1}$, so using Lemma 5.2.9 once again we obtain that

$$A_{j+1}^{\alpha_{j+1}+1} \leq C\,(\alpha_j + 1)^{m-1} \big(A_j^{k(\alpha_j + m) - m + 1} + A_j^{k(\alpha_j + m)} \big) (j+1)^2.$$

Since $Q \geq 1$ in $B_{3\rho} \times [a, b]$

$$A_{j+1} \leq \big(C\,(\alpha_j + 1)^{m-1} \big)^{1/(\alpha_{j+1}+1)} \cdot A_j (j+1)^{2/(\alpha_{j+1}+1)}$$

which easily gives that

$$A_\infty = \|Q\|_{L^\infty(R)} \leq C\,A_0$$

and the lemma is proved. □

Lemma 5.2.12. *For any $I = (a, b)$ and D such that $D \times I = \Omega_0 \subset\subset \Omega$, $w(x, t)$ and $\int_a^t u^m(x, s)\, ds$ are bounded in Ω_0. Furthermore if $(x, t) \in \Omega_0$*

$$w(x, t) = Gv_a(x) - \int_a^t u^m(x, s)\, ds + h(x, t). \tag{5.2.11}$$

where $\Delta h = 0$ and h is bounded in Ω_0.

Proof. Define

$$h(x,t)=\iint_{\partial D\times[a,t]}\frac{\partial G}{\partial n}(x,\xi)d\mu(\xi)$$

and observe that the relation (5.2.11) follows from the equations $\partial q/\partial t+u^m=0$ with $q=w-h$. Since

$$\sup_{t\in I}\int_D h(x,t)\,dx<\infty$$

by Harnack's inequality

$$h(x,t)\le C\quad \text{for any } (x,t)\in K\times I \text{ with } K\subset\subset D.$$

From Lemma 5.2.10 it follows that

$$\sup_{(x,t)\in K\times I}\int_a^t u^m(x,s)\,ds<\infty$$

which yields the lemma. □

Notice that if $D\subset\subset\tilde{D}$ with $\partial\tilde{D}$ smooth, and $G'_{\tilde{D}}$ denotes the Green's function of $\tilde{D}$, by the maximum principle $G'_{\tilde{D}}\ge G_D$. Therefore $w\le G'u$ in Ω_0.

Next we study the regularity of w in Sobolev spaces. We recall that $H_0^1(D)$ denotes the completion of $C_0^\infty(D)$ in the norm

$$\|\eta\|_{H_0^1}=\left(\int_D|\nabla\eta|^2\,dx\right)^{1/2}.$$

First we observe that if $D\subseteq\mathbb{R}^n$ is a bounded domain such that $D\times(a,b)\subset\subset\Omega$, then $w=Gu=Gv_t$ satisfies

$$w(\cdot,t)\in H_0^1(D)$$

for all $t\in[a,b]$. This follows from the boundedness of w since

$$\int_D|\nabla w(x,t)|^2\,dx=\int w\,dv_t<\infty.$$

where v_t is the trace defined in the Step 1. For functions $f\in H_1^0(D)$ we shall use the notation

$$\mathcal{E}(f)=\int_D|\nabla f|^2\,dx.$$

We want now to show that $u\in L_{\text{loc}}^{m+1}(\Omega)$.

Theorem 5.2.13. *If $u\in L_{\text{loc}}^m$, $m>1$, with $u\ge 0$ a weak solution of the porous medium equation*

$$\frac{\partial u}{\partial t}=\Delta u^m$$

in Ω. Then $u\in L_{\text{loc}}^{m+1}(\Omega)$.

Proof. Let $\Omega_0 = D \times (a, b)$ such that $\Omega_0 \subset\subset \Omega$. Let K be a compact subset of D. Denote by $\nu = \nu_a =$ the trace of u at $t = a$. Let

$$v(x,t) = G\nu(x) - \int_a^t u^m(x,s)ds.$$

Pick $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$ with $\varphi(x,t) = \theta(x)\,\eta(t)$, $\theta \geq 0$, $\eta \geq 0$ and $\int \eta = \int \theta = 1$. Let

$$\lambda_\varepsilon v(x,t) = \iint \varphi_\varepsilon(\xi,\tau)\, v(x-\xi, t-\tau)\, d\xi d\tau \tag{5.2.12}$$

where $\varphi_\varepsilon(\xi,\tau) = \varepsilon^{-1-n} \cdot \varphi(x/\varepsilon, t/\varepsilon)$. Thus

$$\frac{\partial \lambda_\varepsilon v}{\partial t} = -\lambda_\varepsilon u^m \leq 0$$

and

$$\Delta \lambda_\varepsilon v = -\lambda_\varepsilon u \leq 0.$$

From Lemma 5.2.2 we obtain that

$$\iint_{K\times(a,b)} \frac{\partial v}{\partial t}\, \Delta v\, dxdt = \iint_{K\times(a,b)} (\lambda_\varepsilon u^m)(\lambda_\varepsilon u)\, dxdt < C$$

where C is independent of ε. By Fatou's Lemma we finally conclude that

$$\iint_{K\times(a,b)} u^{m+1}(x,t)\, dxdt < C$$

which is the desired result. □

Let $B = B(x,\tau) = \{x : |x| < r\}$. Assume that $\Omega_0 = B \times (a,b) \subset\subset \Omega$. We recall that

$$\int_B \psi(x,b)\, d\nu_b = \int_B \psi(x,a)\, d\nu_a + \iint_{\Omega_0} \left\{ u^m \Delta\psi + u\,\frac{\partial \psi}{\partial t} \right\} dxdt + \iint_{\partial B\times[a,b]} \frac{\partial \psi}{\partial n}\, d\mu$$

for any $\psi \in C_0^\infty(\mathbb{R}^{n+1})$ with $\psi \equiv 0$ on $\partial B \times \mathbb{R}$. Also we recall that ν_τ and μ are weakly continuous measures, and that $w = Gu$ (Green's function of u in B) is bounded.

To complete the proof of Theorem 5.2.1 we need the following lemma.

Lemma 5.2.14. *There exists a continuous weak solution $u^* \geq 0$ of the porous medium equation $\partial u^*/\partial t = \Delta(u^*)^m$ in $\Omega_0 = B \times (a,b)$ such that $u = u^*$ a.e. in Ω_0.*

Proof. Let ε_0 be a small positive constant. For $\varepsilon \in (0, \varepsilon_0)$ denote $B_\varepsilon = B(x_0, r-\varepsilon)$, $G_\varepsilon =$ the Green's function of B_ε, $w_\varepsilon = G_\varepsilon u_\varepsilon$, $\Omega_{0_\varepsilon} = B_\varepsilon \times (a,b)$ and $u_\varepsilon = u_\varepsilon(x,t) = \lambda_\varepsilon u(x,t)$ (where λ_ε was defined in (5.2.12)).

Denote by u_ε^* the continuous weak solution of the porous medium equation in Ω_{0_ε} with initial values

$$u_\varepsilon^* = \begin{cases} u_\varepsilon & \text{on } B_\varepsilon \times \{a\}, \\ (\lambda_\varepsilon u^m))^{1/m} & \text{on } \partial B_\varepsilon \times [a,b]. \end{cases}$$

Let $\delta_\varepsilon(x) = \operatorname{dist}(x, \partial B_\varepsilon)$. We claim that there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$

$$\sup_{t\in(a,b)} \int_{B_\varepsilon} \delta_\varepsilon(x) u_\varepsilon^*(x,t)\,dx \le C \quad \text{and} \quad \iint_{\Omega_{0_\varepsilon}} (u_\varepsilon^*)^m\,dxdt \le C. \tag{5.2.13}$$

To prove this claim we use the weak continuity of the measure of the boundary side respect to the radius R. For $\psi \in C_0^\infty(\mathbb{R})$ with $\psi \equiv 0$ on ∂B, define

$$\psi_\varepsilon(x) = \psi\left(\frac{(x-x_0)}{r-\varepsilon} + x_0\right).$$

Then $\psi_\varepsilon \equiv 0$ on ∂B_ε and

$$\lim_{\varepsilon\downarrow 0} \int_{B_\varepsilon} \psi(x) u_\varepsilon^*(x,a)dx = \int_B \psi(x)\,dv_a$$

by the weakly continuity of the measure v_τ. Again by the weakly continuity

$$\lim_{\varepsilon\to 0} \iint_{\partial B_\varepsilon\times(a,b)} \frac{\partial\psi_\varepsilon}{\partial\eta_\varepsilon}\,(u_\varepsilon^*)^m\,d\sigma dt \equiv \iint_{\partial B\times(a,b)} \frac{\partial\psi}{\partial n}\,d\mu$$

which proves the claim.

By our definition of weak solution (using the above claim) we have

$$-\iint_{\partial B_\varepsilon\times(a,b)} \frac{\partial\psi_\varepsilon}{\partial n_\varepsilon}(u_\varepsilon^*)^m = \int_{B_\varepsilon} \psi_\varepsilon(x,a) u_\varepsilon^*(x,a)\,dx + \iint_{\Omega_{0_\varepsilon}} \left((u_\varepsilon^*)^m \Delta\psi_\varepsilon + u_\varepsilon^* \frac{\partial\psi_\varepsilon}{\partial t}\right) - \int_{B_\varepsilon} \psi_\varepsilon(x,b)\,u_\varepsilon^*(x,b)\,dx.$$

Claim. There exists $u^* \in C(\Omega_0)$ non-negative weak solution of the porous medium equation (no boundary value problem) such that $u_\varepsilon^* \to u^*$ uniformly on compact subsets of Ω_0. Moreover,

$$\int_B \psi(x,b) u^*(x,b)\,dx = \int_B \psi(x,a)\,dv_a + \int_{\Omega_0} \left\{(u^*)^m \Delta\psi + \frac{\partial\psi}{\partial t}\,u^*\right\} dxdt + \iint_{\partial B\times(a,b)} \frac{\partial\psi}{\partial n}\,d\mu.$$

Indeed, let $h_\varepsilon(x,t)$ be the solution of

$$\begin{aligned} \Delta h_\varepsilon &= 0 && \text{in } \Omega_{0_\varepsilon}, \\ h_\varepsilon|_{\partial B_\varepsilon} &= \lambda_\varepsilon u^m && \text{on } \partial B_\varepsilon \times (a,b), \end{aligned}$$

and let $w_\varepsilon^* = G_\varepsilon u_\varepsilon^*$. Then

$$\frac{\partial w_\varepsilon}{\partial t} = \frac{\partial G_\varepsilon(\lambda_\varepsilon u)}{\partial t} = G_\varepsilon \lambda_\varepsilon \left(\frac{\partial u}{\partial t}\right) = G_\varepsilon \Delta(\lambda_\varepsilon u^m) = h_\varepsilon - \lambda_\varepsilon u^m$$

and

$$\frac{\partial w_\varepsilon^*}{\partial t} = G_\varepsilon \frac{\partial u_\varepsilon^*}{\partial t} = G_\varepsilon \Delta (u_\varepsilon^*)^m = h_\varepsilon - (u_\varepsilon^*)^m.$$

Define $E_\varepsilon = w_\varepsilon^* - w_\varepsilon$ and

$$l_\varepsilon(t) = \mathcal{E}(E_\varepsilon(\,\cdot\,,t)) = \int_{B_\varepsilon} |\nabla E_\varepsilon|^2 \, dx.$$

Notice that

$$E_\varepsilon(\,\cdot\,,a) \equiv 0.$$

Therefore $l_\varepsilon(a) = 0$. Also

$$l_\varepsilon(t) = \int_{B_\varepsilon} \left|\nabla(w_\varepsilon^* - w_\varepsilon)\right|^2 dx = \int_{B_\varepsilon} (w_\varepsilon^* - w_\varepsilon)\, \Delta(w_\varepsilon^* - w_\varepsilon)\, dx.$$

Thus

$$\begin{aligned} \frac{d}{dt} l_\varepsilon(t) &= -\int_{B_\varepsilon} \frac{\partial w_\varepsilon^*}{\partial t}\, \Delta(w_\varepsilon^* - w_\varepsilon) - \int_{B_\varepsilon} (w_\varepsilon^* - w_\varepsilon)\, \Delta \partial_t (w_\varepsilon^* - w_\varepsilon) \\ &= -2 \int_{B_\varepsilon} \frac{\partial E_\varepsilon}{\partial t}\, \Delta E_\varepsilon. \end{aligned}$$

Integration in time leads to

$$\iint_{\Omega_{0_\varepsilon}} \frac{\partial E_\varepsilon}{\partial t}\, \Delta E_\varepsilon \, dxdt = -\frac{1}{2} \int_a^b \frac{\partial l_\varepsilon}{\partial t} \le 0 \tag{5.2.14}$$

since $l_\varepsilon(a) = 0$ and $l_\varepsilon(\,\cdot\,) \ge 0$.

Observe that $\lambda_\varepsilon u \to u$ in $L^{m+1}(\Omega_0)$ since we have already proved that $u \in L^{m+1}(\Omega_0)$. Define $L_\varepsilon^{m+1} \equiv \iint_{\Omega_{0_\varepsilon}} (u_\varepsilon^*)^{(m+1)} dxdt$. Then L_ε^{m+1} is uniformly bounded in $\varepsilon \in (0, \varepsilon_0)$. To see this we compute

$$\begin{aligned}
L_\varepsilon^{m+1} &= \iint_{\Omega_{0_\varepsilon}} (u_\varepsilon^*)\,(u_\varepsilon^*)^m \\
&= \iint_{\Omega_{0_\varepsilon}} u_\varepsilon^*\,(h_\varepsilon - \frac{\partial w_\varepsilon^*}{\partial t}) \\
&= \iint_{\Omega_{0_\varepsilon}} u_\varepsilon^*\,h_\varepsilon + \iint_{\Omega_{0_\varepsilon}} \Delta w_\varepsilon^*\,\frac{\partial w_\varepsilon^*}{\partial t} \\
&= \iint_{\Omega_{0_\varepsilon}} u_\varepsilon^*\,\lambda_\varepsilon u^m + \iint_{\Omega_{0_\varepsilon}} u_\varepsilon^*\,\frac{\partial w_\varepsilon}{\partial t} + \iint_{\Omega_{0_\varepsilon}} \Delta w_\varepsilon^*\,\frac{\partial w_\varepsilon^*}{\partial t} \\
&= \iint u_\varepsilon^*\lambda_\varepsilon u^m + \iint \left(-\Delta w_\varepsilon^*\,\frac{\partial w_\varepsilon}{\partial t} + \Delta w_\varepsilon^*\,\frac{\partial w_\varepsilon^*}{\partial t}\right).
\end{aligned}$$

Since

$$\iint \left(\frac{\partial w_\varepsilon^*}{\partial t} - \frac{\partial w_\varepsilon}{\partial t}\right)\left(\Delta w_\varepsilon^* - \Delta w_\varepsilon\right) \le 0$$

(proved above in (5.2.14)) the last expression is bounded by

$$\begin{aligned}
&\iint u_\varepsilon^*\,\lambda_\varepsilon u^m + \iint \left(-\frac{\partial w_\varepsilon}{\partial t}\,\Delta w_\varepsilon + \frac{\partial w_\varepsilon^*}{\partial t}\,\Delta w_\varepsilon\right) \\
&\qquad = \iint u_\varepsilon^*\lambda_\varepsilon u^m + \iint u_\varepsilon\left((u_\varepsilon^*)^m - \lambda_\varepsilon u^m\right) \\
&\qquad \le \iint u_\varepsilon^*\lambda_\varepsilon u^m + \iint u_\varepsilon (u_\varepsilon^*)^m.
\end{aligned}$$

Thus

$$L_\varepsilon^{m+1} \le C\,L_\varepsilon + C\,L_\varepsilon^m$$

and therefore L_ε is bounded uniformly in ε.

Thus we can assume that

$$(u_\varepsilon^*)^m \to u^* \quad \text{weakly in } L^{(m+1)/m}(\Omega_0)$$

and

$$u_\varepsilon^* \to u^* \quad \text{weakly in } L^{m+1}(\Omega_0).$$

Now for $T \in (a, b)$ we have

$$\begin{aligned}
0 \le l_\varepsilon(T) &\equiv -2 \iint_{\Omega_{0_\varepsilon}} \frac{\partial E_\varepsilon}{\partial t}\,\Delta E_\varepsilon \\
&= 2 \iint_{\Omega_{0_\varepsilon}} (u_\varepsilon - u_\varepsilon^*)\,(\lambda_\varepsilon u^m - (u_\varepsilon^*)^m) \\
&= 2 \iint \lambda_\varepsilon u\,\lambda_\varepsilon u^m - 2 \iint u_\varepsilon^*\lambda_\varepsilon u^m - 2 \iint u_\varepsilon (u_\varepsilon^*)^m + 2 \iint u_\varepsilon^*\,(u_\varepsilon^*)^m.
\end{aligned}$$

Hence letting $\varepsilon \to 0$ we obtain

$$0 \leq l_\varepsilon(T) \leq 0$$

from which the lemma follows. □

Theorem 5.2.1 is an immediate consequence of the previous lemma. □

5.3 Further results and open problems

1. The results in this chapter extend, without difficulty, to general $\varphi \in \mathcal{S}_a$ (slow diffusion) which are in addition convex. It would be interesting to establish this for general $\varphi \in \mathcal{S}_a$.

2. One could ask about corresponding results for variable sign solutions with u^m replaced by $|u|^{m-1}u$. In light of the counterexamples of Serrin [119], it is not clear whether the corresponding results hold for variable sign solutions.

3. The corresponding questions can be formulated for weak solutions in a cylinder and weak solutions respectively in the fast diffusion case with $m_1 < m < 1$ and $m_c < m < 1$, as defined in Chapter 4. This is an interesting and challenging question.

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